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TIME-DEPENDENT DESCRIPTORS FOR THE POISSON QUEUE

by

RHODA HORNKOHL BARR

June 1973

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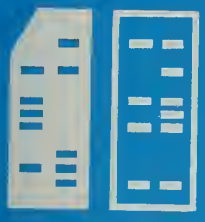

DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN · URBANA, ILLINOIS

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TIME-DEPENDENT DESCRIPTORS FOR THE POISSON QUEUE

BY

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June 1973

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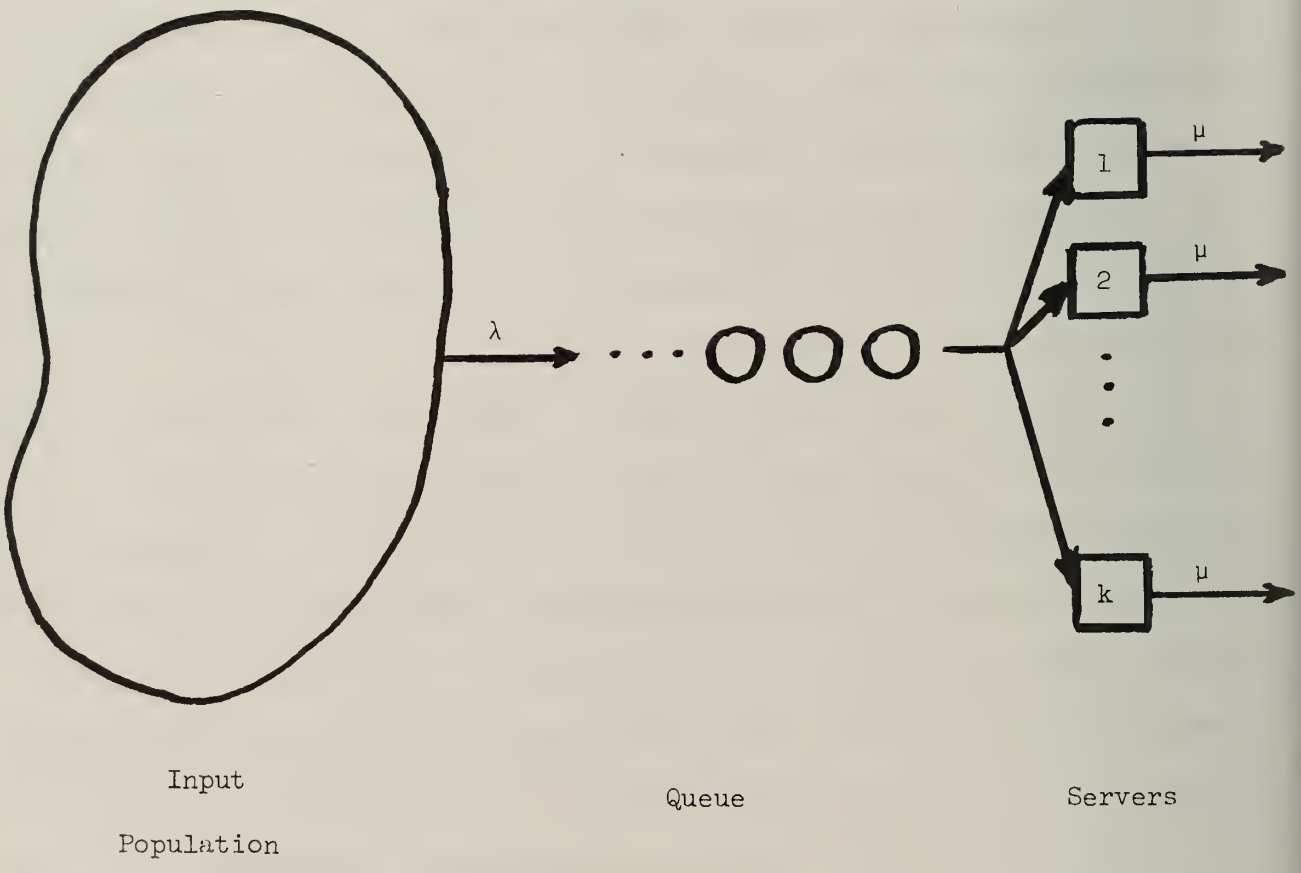
1. INTRODUCTION

Man has been searching for ways to avoid waiting in line for thousands of years. In ancient Rome the man who didn't want to wait all day in the breadline elbowed his way to the head of the line. Today the modern shopper confronted with six checkout counters tries to guess which line will move most quickly. The application of analytical tools such as modeling and simulation to queueing problems represents a small triumph for civilization, as we replace force and guesswork by reason in our attempts to reduce waiting times.

One of the first applications of queueing theory was to the problem of scheduling appointments with physicians so that patients' waiting time is kept to a minimum while physicians are kept busy [8, 10]. Other obvious applications have been to customer queues in banks and supermarkets. The advent of third-generation computers, however, has provided the major impetus for research in this area. The reason for this is hardly obscure. Few human beings can consistently command more than $100.00 per hour for their services. While accounting methods vary, the use of a large- or medium-scale computing system costs on the order of $1.00 per second. Thus a solution to the problem of keeping a computer from waiting idle for its next task becomes economically, if not aesthetically, more important than problems related to the convenience of people who don't like to wait in lines. Happily, results derived from the study of jobs waiting for service from a computer are also applicable to the study of customers waiting for a doctor or bank teller, because queueing analysis is largely independent of any particular application.

The first step in analyzing a queueing system is usually to deduce the nature of arrivals and service in the system, perhaps make some simplifying assumptions about it and then construct a mathematical model of the system. In this paper we are concerned with Poisson queueing systems which are schematized in Figure 1 and have the following characteristics:



The k-Server Poisson Queueing System

Figure 1

1. infinite arrival population
2. exponentially distributed arrivals with average arrival rate $\lambda > 0$
3. k homogeneous servers in parallel
4. service discipline is "first-come, first-served"
5. exponentially distributed service with average service rate $\mu > 0$
6. infinite queue length
7. initially there are i units in the system awaiting service, $i \geq 0$.

For this system it is known that the probability of n arrivals in a time interval of length Δt is $\frac{(\lambda\Delta t)^n}{n!} e^{-\lambda\Delta t}$ and the probability of n services by a single server in an interval of the same duration is $\frac{(\mu\Delta t)^n}{n!} e^{-\mu\Delta t}$ [6].

We denote by $P_n(t)$ the probability that at a given time t there are exactly n units in the system, both waiting and in service. Where there is one server the system has been described by the following set of differential difference equations:

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \tag{1.1}$$

$$P_n'(t) = -(\lambda+\mu)P_0(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), \quad n \geq 1. \tag{1.2}$$

The differential difference equations which describe the system with $k > 1$ servers are:

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \tag{1.3}$$

$$P_n'(t) = -(\lambda+n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t), \quad 1 \leq n < k \tag{1.4}$$

$$P_n'(t) = -(\lambda+k\mu)P_n(t) + \lambda P_{n-1}(t) + k\mu P_{n+1}(t), \quad n \geq k. \tag{1.5}$$

Some specific systems which can be described by this model are single-processor computing systems such as the Univac 1106 or multi-processor systems which select tasks from a single queue, such as the Univac 1108 and Univac 1110. Some localized

network computers with negligible transmission times, such as the Distributed Computer System [5] and the Collins C-System [3], can also be treated as multiprocessor systems.

Another application for the model is to systems which reach one steady state but don't remain in it. A change in arrival rate or the number of servers, for example, would cause a transition from one steady state to another. The model could be used to describe this transition.

Having constructed a mathematical model of a queueing system, one would then like to use it to find out how the system operates. Some of the questions one might ask are: How many units are likely to be waiting for service at a given time? How often will there be no units at all waiting to be served? How often will the servers be left idle? With what probability will there be exactly n units in the system (i.e., what is $P_n(t)$)? Does the system ever reach a point in time after which the values $P_n(t)$ are static?

When the answer to this last questions is yes, the system is said to reach steady state. This is known to occur if and only if $\lambda < k\mu$. Since the system is no longer changing with respect to time, $P_n'(t) = 0$. Moreover, as $t \to \infty$, $P_n(t)$ takes on the static value that no longer depends on t, denoted P_n. The well-known solution to the steady state problem is

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) \tag{1.6}$$

when there is one server, and

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, \quad 0 \leq n \leq k \tag{1.7}$$

$$P_n = \frac{1}{k^{n-k}k!}\left(\frac{\lambda}{\mu}\right)^n P_0, \quad n \geq k \tag{1.8}$$

$$P_0 = \left\{ \sum_{n=0}^{k-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{k!} \left(\frac{\lambda}{\mu}\right)^k \frac{k\mu}{k\mu-\lambda} \right\}^{-1} \tag{1.9}$$

where there are $k > 1$ servers.

Many other descriptors have been determined for the queueing system at steady state. The expected number of units in the system is defined as

$$E_n = \sum_{n=0}^{\infty} nP_n \, . \tag{1.10}$$

In the one-server case this is evaluated as $\frac{\lambda}{\mu-\lambda}$. In the many-server case it is evaluated as

$$\frac{(\lambda/\mu)^k \lambda\mu P_0}{(k-1)!(k\mu-\lambda)^2} + \frac{\lambda}{\mu} \, . \tag{1.11}$$

Sometimes one is interested only in the number of units in the queue rather than the number of units in the system, which includes both units in service and those waiting in the queue. The probability of having exactly m units in the queue at steady state, denoted P_{mq}, is defined as

$$P_{0q} = \sum_{n=0}^{k} P_n \tag{1.12}$$

$$P_{mq} = P_{m+k}, \; m \geq 1. \tag{1.13}$$

Then the expected number of units in the queue is defined as

$$E_m = \sum_{m=0}^{\infty} mP_{mq} \, . \tag{1.14}$$

In the one-server case this is evaluated as

$$\frac{(\lambda/\mu)^2}{1-(\lambda/\mu)}, \tag{1.15}$$

and in the many-server case it is

$$\frac{(\lambda/\mu)^k \lambda\mu P_0}{(k-1)!(k\mu-\lambda)^2}. \tag{1.16}$$

When studying a queueing system with more than one server it may be useful to know something about how often the servers are idle. If exactly n servers are idle then there are k - n units in the system and the expected number of idle servers is defined as

$$E_I = \sum_{n=0}^{k} nP_{k-n}$$

or equivalently

$$\sum_{n=0}^{k-1} (k-n)P_n. \tag{1.17}$$

This is evaluated as $k - \frac{\lambda}{\mu}$ [12].

Thus at steady state we know how many units to expect in the queue and in the system (both waiting in queue and in service), and how many servers will be idle.

A steady state analysis is most useful when the system is known to reach steady state quickly and outside interference will not cause it to be jolted out of steady state. A sensitivity analysis of the Poisson queueing system shows that the smaller the ratio $\lambda/k\mu$ the faster the system reaches steady state [14]. For such systems a transient analysis supplies little extra information, but only the transient analysis can show precisely how fast the system reaches steady state and

that it actually stays there. Moreover, a serious shortcoming of steady state analysis is that it cannot be applied at all to the many systems which never reach steady state. For these systems only a time-dependent analysis is possible.

Many time-dependent, or transient, solutions to the differential difference equations modelling the Poisson queueing system have been discussed in the literature. First Bailey [1] determined an exact analytic expression for $\frac{\partial P_n(t)}{\partial t}$ in the one-server case. Later Saaty [12] and Cox and Smith [4] determined $P_n(t)$ in the one-server case, and Saaty [13] derived an expression for the Laplace transform of $P_n(t)$ in the many-server case. Unfortunately, all of these expressions are far more complicated than the steady state solutions and it is difficult to obtain really useful information from them. This problem will be discussed further in Chapter II.

Several other descriptors for the system at steady state were given above. Just as at steady state, one could ask how many units in the system, units in the queue, or idle servers can be expected at a given time. We define time-dependent functions which are analogous to these steady state descriptors. The expected number of units in the system at time t is

$$E_n(t) = \sum_{n=0}^{\infty} nP_n(t). \tag{1.18}$$

The probability of having m units in the queue at time t is

$$P_{0q}(t) = \sum_{n=0}^{k} P_n(t) \tag{1.19}$$

$$P_{mq}(t) = P_{m+k}(t). \tag{1.20}$$

Then the expected number of units in the queue at time t is

$$E_m(t) = \sum_{m=0}^{\infty} mP_{mq}(t). \tag{1.21}$$

Finally, the expected number of idle servers at time t when there are more than one server is

$$E_I(t) = \sum_{n=0}^{k-1} (k-n)P_n(t). \tag{1.22}$$

In Chapter III we determine analytically the Laplace transforms of the functions $E_n(t)$, $E_m(t)$ and $E_I(t)$ for the cases $k = 1$ and $k = 2$. We further show how our method can be applied to achieve similar results when $k > 2$. In Chapter IV these functions are inverted numerically for some selected values of the parameters λ, μ, k and i. We produce tables of values for $E_n(t)$, $E_m(t)$ and $E_I(t)$ in these cases. Finally, in Chapter V we discuss some implications of this work and suggest areas for future research.

2. TRANSIENT SOLUTIONS TO THE POISSON QUEUE

In this chapter we discuss some solutions for $P_n(t)$ in the one-server and many-server cases. We are concerned mainly with preparing a foundation for the work presented in Chapter III.

Saaty's [12] complete solution to the single server Poisson queue is outlined in Appendix I. He finds that

$$P_n(t) = e^{-(\lambda+\mu)t}\left\{\left(\sqrt{\frac{\mu}{\lambda}}\right)^{i-n} I_{n-i} + \left(\sqrt{\frac{\mu}{\lambda}}\right)^{i-n+1} I_{i+n+1} + \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) \sum_{j=n+i+2}^{\infty} \left(\sqrt{\frac{\mu}{\lambda}}\right)^j I_j \right\} \tag{2.1}$$

where I_j is the modified Bessel function of the first kind with argument $2\sqrt{\lambda\mu}\,t$.

Many other solutions have been given for this problem. Bailey [1] shows that

$$\frac{\partial P_n(t)}{\partial t} = \left(\sqrt{\frac{\mu}{\lambda}}\right)^{i+n} e^{-(\lambda+\mu)t}\left\{-(\lambda+\mu)I_{i-n} + \sqrt{\lambda\mu}\, I_{i-n-1} + \sqrt{\lambda\mu}\, I_{i-n+1} + \lambda I_{i+n+2} - 2\sqrt{\lambda\mu}\, I_{i+n+1} + \mu I_{i+n}\right\} \tag{2.2}$$

where I_j is the modified Bessel function of the first kind with argument $2\sqrt{\lambda\mu}\,t$.

Cox and Smith [4] formulate the problem somewhat differently. The time unit is chosen so that the parameter of the service distribution $\mu = 1$. The corresponding parameter for the arrival distribution is ρ. Then for the one server case the differential difference equations become

$$P_0'(t) = -\rho P_0(t) + P_1(t) \tag{2.3}$$

$$P_n'(t) = \rho P_{n-1}(t) - (1+\rho)P_n(t) + P_{n+1}(t), \quad n > 0. \tag{2.4}$$

Now it is assumed that (2.4) holds for all integers n. If a solution can then be found satisfying the additional constraint $P_0(t) = \rho P_{-1}(t)$, (2.3) is satisfied and it is the required solution. It is found to be

$$P_n(t) = \rho^{\frac{1}{2}(n-1)} e^{-(1+\rho)t} \left\{ I_{i-n} + \sqrt{\rho}\, I_{n+i+1} + (1-\rho) \sum_{j=1}^{\infty} \rho^{-\frac{1}{2}-\frac{1}{2}j} I_{n+i+1+j} \right\}. \tag{2.5}$$

This is completely analogous to the solution given by Saaty.

It can be proved [12] that

$$\sum_{n=0}^{\infty} P_n(t) = 1. \tag{2.6}$$

This is a stronger result than the steady state solution. As shown in Appendix I, in the transient solution Rouché's theorem is used to determine $P_0^*(s)$. But at steady state it must be assumed that

$$\sum_{n=0}^{\infty} P_n = 1 \tag{2.7}$$

to determine P_0 and complete the solution.

It can also be proved [12] that when $\lambda < \mu$

$$\lim_{t\to\infty} P_n(t) = P_n. \tag{2.8}$$

Thus the functions $P_n(t)$ describe the approach of the system to its steady state. When $\lambda \geq \mu$, units arrive in the system faster than they can be serviced and the

length of the queue increases without bound. This is why no steady state analysis can be made when $\lambda \geq \mu$. In this case, the functions $P_n(t)$ describe the indefinite growth of the system [4].

Theoretically the functions $P_n(t)$ could be used to determine other properties of the Poisson queueing system. In practice the task is formidable. Each function is an infinite sum and each term in the sum includes a Bessel function, which is defined as an infinite sum in Appendix I. Thus the usefulness of these expressions for $P_n(t)$ is limited.

Next we sketch the solution given by Saaty [13] for the many-server Poisson queue. The differential difference equations which model this problem are given by (1.3), (1.4), and (1.5). As in the one-server case, we multiply each equation by z^{n+1}, sum the resulting equations to get an expression in terms of the generating function, take Laplace transforms, and impose the initial condition that $P_n(0) = \delta_{i0}$. We then find

$$\prod^*(z,s) = \frac{z^{i+1} - \mu(1-z)\sum_{n=0}^{k-1}(k-n)z^n P_n^*(s)}{sz - (1-z)(k\mu-\lambda z)} . \tag{2.9}$$

It can be shown that $|z| \leq 1$ and Re $s > 0$ are sufficient conditions for convergence of $\prod^*(z,s)$ in the k-server case using the same proof as that for the single server case in Appendix I. Then Rouché's theorem can be applied to determine that

$$\sum_{n=0}^{k-1}(k-n)\alpha_2^n P_n^*(s) = \frac{\alpha_2^{i+1}}{\mu(1-\alpha_2)} \tag{2.10}$$

where

$$\alpha_1 = \frac{\lambda+k\mu+s+\sqrt{(\lambda+k\mu+s)^2-4k\lambda\mu}}{2\lambda}$$

and

$$\alpha_2 = \frac{\lambda+k\mu+s-\sqrt{(\lambda+k\mu+s)^2-4k\lambda\mu}}{2\lambda} . \tag{2.11}$$

It is shown that for $n \geq k$

$$P_n^*(s) = \frac{1}{(\alpha_1-\alpha_2)\lambda}\left\{(\lambda+s)P_0^*(s)\left(\frac{1}{\alpha_2^n}-\frac{1}{\alpha_1^n}\right) - \lambda P_0^*(s)\left(\frac{1}{\alpha_2^{n-1}}-\frac{1}{\alpha_1^{n-1}}\right)\right.$$
$$\left.-\left(\frac{1}{\alpha_2^{n-i}}-\frac{1}{\alpha_1^{n-i}}\right)\right\}. \tag{2.12}$$

To solve for $P_n^*(s)$ when $n \leq k-1$ requires the Laplace transforms of (1.3) and (1.4). They are

$$(\lambda+s)P_0^*(s) - \mu P_1^*(s) = \delta_{io} \tag{2.13}$$

$$(\lambda+s+n\mu)P_n^*(s) - \lambda P_{n-1}^*(s) - (n+1)\mu P_{n+1}^*(s) = \delta_{in},\ 1 \leq n < k. \tag{2.14}$$

In [13] the problem is restricted so that $i \geq k-1$, which implies that $\delta_{io} = 0$. Then for $k = 2$ (2.10) and (2.13) are solved simultaneously, giving

$$P_0^* = \frac{\alpha_2^{i+1}}{(1-\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2)}$$

and

$$P_1^*(s) = \frac{(s+\lambda)\alpha_2}{\mu(1-\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2)} . \tag{2.15}$$

For $k \geq 3$, (2.10), (2.13), and the first $k - 2$ equations of (2.14) can be solved simultaneously to determine $P_n^*(s)$, $0 \leq n \leq k - 1$.

For the case $k = 2$, we extend the results given to hold for any i and find that

$$P_0^*(s) = \frac{\alpha_2^{i+1} + \delta_{i0}\alpha_2(1-\alpha_2)}{(1-\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2)}$$

and

$$P_1^*(s) = \frac{(\lambda+s)\alpha_2^{i+1} - 2\mu\delta_{i0}(1-\alpha_2)}{\mu(1-\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2)} . \tag{2.16}$$

The solution to the many-server queue is known only in terms of its Laplace transform. The final value theorem for Laplace transforms, which states that

$$\lim_{s\to 0} sf^*(s) = \lim_{t\to\infty} f(t) \quad [11] \tag{2.17}$$

can be used to prove that when $\lambda < k\mu$

$$\lim_{t\to\infty} P_n(t) = P_n. \tag{2.18}$$

Thus when $\lambda < k\mu$ the functions $P_n^*(s)$ in some sense describe the system's approach to steady state.

All of these transient solutions to the Poisson queue are important because they describe the system at any point in time, and they provide information about systems which don't reach steady state. We would next like to use the results of solving the differential difference equations modelling the Poisson queue to determine some time-dependent descriptors for Poisson queueing systems. Because of the complexity of the functions $P_n(t)$ and $P_n^*(s)$ which have been determined, we choose to avoid using them directly by concentrating instead on some of Saaty's intermediate results in finding $P_n(t)$ and $P_n^*(s)$.

3. LAPLACE TRANSFORMS OF TRANSIENT SYSTEM DESCRIPTORS

In this chapter we determine analytically the Laplace transforms of the transient system descriptors $E_n(t)$, $E_m(t)$ and $E_I(t)$ which were defined in the introduction. We begin with the expected number in the system, which includes both units in service and those waiting in the queue. This is given by (1.18). The straight-forward manner of evaluating this would be to substitute an expression for $P_n(t)$ from Chapter II. But $P_n(t)$ was shown to be a double infinite sum, so $E_n(t)$ would be a triple infinite sum which we have no desire to evaluate. Instead, we recall that the generating function of $P_n(t)$ is defined in Appendix I as

$$\Pi(z,t) = \sum_{n=0}^{\infty} z^n P_n(t). \tag{3.1}$$

Formally, we can say that

$$E_n(t) = \frac{\partial}{\partial z} \{\Pi(z,t)\}\Big|_{z=1}. \tag{3.2}$$

This is of limited value because we have no nice expression for $\Pi(z,t)$. But we do know from equations (I.4) and (I.5) of Appendix I that in the one-server case

$$\prod^*(z,s) = \frac{z^{i+1} - (1-z)\alpha_2^{i+1}/(1-\alpha_2)}{sz - (1-z)(\mu-\lambda z)}. \tag{3.3}$$

If we can prove that the proper convergence conditions hold, we will be able to conclude that

$$E_n^*(s) = \frac{\partial}{\partial z} \{\Pi^*(z,s)\} \Big|_{z=1}.$$

Now,

$$E_n^*(s) = \int_0^\infty E_n(t)e^{-st}dt$$

$$= \int_0^\infty \frac{\partial}{\partial z}\{\Pi(z,t)\}\Big|_{z=1} e^{-st}dt. \tag{3.4}$$

Also,

$$\frac{\partial}{\partial z}\{\Pi^*(z,s)\}\Big|_{z=1} = \frac{\partial}{\partial z}\left\{\int_0^\infty \Pi(z,t)e^{-st}dt\right\}\Big|_{z=1}. \tag{3.5}$$

We can conclude that

$$E_n^*(s) = \frac{\partial}{\partial z}\{\Pi^*(z,s)\}\Big|_{z=1} \tag{3.6}$$

if

$$\int_0^\infty \frac{\partial}{\partial z}\{\Pi(z,t)\}\Big|_{z=1} e^{-st}dt$$

is uniformly convergent, because then

$$\int_0^\infty \frac{\partial}{\partial z}\{\Pi(z,t)\}\Big|_{z=1} e^{-st}dt = \frac{\partial}{\partial z}\left\{\int_0^\infty \Pi(z,t)e^{-st}dt\right\}\Big|_{z=1}. \tag{3.7}$$

We do this by establishing an upper bound $E_n(t)$. Because arrivals obey a Poisson distribution, the probability of j arrivals in a time interval of length t is $\frac{(\lambda t)^j}{j!}e^{-\lambda t}$. We only consider the case where t is finite. If t increases without bound either $\lambda < k\mu$ and one can apply steady state analysis, or $\lambda \geq k\mu$ and the expected number in the system increases without bound also. When t is finite, the probability that there are an infinite number of arrivals in the interval

$(0,t)$ is $\lim_{j\to\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} = 0$. Thus there can be only a finite number of arrivals, say j of them where $j \geq 1$, in $(0,t)$. Since there are i units waiting initially in the system $E_n(t) \leq i + j$. Then

$$\int_0^\infty \frac{\partial}{\partial z} \{\Pi(z,t)\} \Big|_{z=1} e^{-st} dt = \int_0^\infty E_n(t) e^{-st} dt \leq \int_0^\infty (i+j) e^{-st} dt = \frac{1}{i+j} \tag{3.8}$$

which is everywhere uniformly convergent. Then by Weierstrauss' M-test for integrals [7]

$$\int_0^\infty \frac{\partial}{\partial z} \{\Pi(z,t)\} \Big|_{z=1} e^{-st} dt$$

is also uniformly convergent.

We can now evaluate $E_n^*(s)$ for Re $s > 0$ as

$$\frac{\partial}{\partial z} \{\Pi^*(z,s)\} \Big|_{z=1}$$

$$= \frac{\partial}{\partial z} \left\{ \frac{z^{i+1} - (1-z)\alpha_2^{i+1}/(1-\alpha_2)}{sz - (1-z)(\mu-\lambda z)} \right\} \Bigg|_{z=1}$$

$$= \frac{i}{s} + \frac{(\lambda-\mu)}{s^2} + \frac{\alpha_2^{i+1}}{s(1-\alpha_2)} . \tag{3.9}$$

The first two terms can be inverted analytically [11] to determine

$$E_n(t) = i + (\lambda-\mu)t + \mathcal{L}^{-1}\left\{ \frac{\alpha_2^{i+1}}{s(1-\alpha_2)} \right\}$$

where $\mathcal{L}^{-1}$ indicates the inverse of the Laplace transform.

We see that the expected number of units in the system depends on i, the number of units in the system initially when t = 0, increases proportionally to λ, the rate at which units arrive in the system, and decreases proportionally to μ, the rate at which units are serviced and leave the system. The last term can be considered a correction term which keeps $E_n(t)$ positive when $\lambda < \mu$.

When there is more than one server, the proof given above suffices to show that

$$E_n^*(s) = \frac{\partial}{\partial z} \{\Pi^*(z,s)\} \Big|_{z=1}.$$

From Saaty's solution to the many-server Poisson queue which was discussed in Chapter II we know that

$$\Pi^*(z,s) = \frac{z^{i+1} + \mu(1-z) \sum_{n=0}^{k-1} (k-n) z^n P_n^*(s)}{sz - (1-z)(k\mu - \lambda z)}.$$

In the two-server case we can substitute the expressions determined for $P_0^*(s)$ and $P_1^*(s)$ and evaluate the partial derivative of $\Pi^*(z,s)$ at z = 1 to find

$$E_n^*(s) = \frac{i}{s} + \frac{\lambda - 2\mu}{s^2} + \frac{(s+\lambda+2\mu)}{(s\alpha_2 + \lambda\alpha_2 + 2\mu)} \frac{\alpha_2^{i+1}}{s(1-\alpha_2)} - \frac{2\mu\delta_{i0}(1-\alpha_2)}{s(s\alpha_2 + \lambda\alpha_2 + 2\mu)}. \tag{3.11}$$

The Laplace inverse of the first two terms [11] is $i + (\lambda - 2\mu)t$. As in the one-server case this shows that the expected number of units in the system depends on i, the number of units waiting in the system initially, increases proportionally to λ, the rate at which units arrive in the system, and decreases proportionally to 2μ, the rate at which units are serviced by the two servers and leave the system.

To determine $E_n^*(s)$ when $k \geq 3$ is rather more complicated algebraically. First, one must solve k equations in the unknowns $P_n^*(s)$, $0 \leq n \leq k - 1$. These functions must be substituted into the expression for $\Pi^*(z,s)$ and then $E_n^*(s)$ can be evaluated in a straightforward manner.

The next descriptor we would like to develop is the expected number of units waiting in the queue, exclusive of those in service. This is given by (1.21). To try to develop an expression for $E_m(t)$ would involve a triple infinite sum, so we seek instead an expression for $E_m^*(s)$, the Laplace transform of $E_m(t)$.

Formally, it can be shown that

$$E_m(t) = E_n(t) - k + \sum_{m=0}^{k-1} (k-m)P_m(t) \tag{3.12}$$

where k is the number of servers in the system. We proved before that $E_n(t)$ is convergent when t is finite and $P_m(t) \leq 1$ for all values of t, so this expression for $E_m(t)$ is valid for finite t. Then by the linearity of the Laplace transform we find

$$E_m^*(s) = E_n^*(s) - \frac{k}{s} + \sum_{m=0}^{k-1} (k-m)P_m^*(s). \tag{3.13}$$

This can be evaluated by substituting for $P_m^*(s)$ the expressions determined in Chapter II and for $E_n^*(s)$ the expressions determined above. In the one-server case we find

$$E_m^*(s) = \frac{i-1}{s} + \frac{\lambda-\mu}{s^2} + \frac{(s+\mu)\alpha_2^{i+1}}{s\mu(1-\alpha_2)}. \tag{3.14}$$

With two servers we find

$$E_m^*(s) = \frac{i-2}{s} + \frac{\lambda-2\mu}{s^2} + \frac{(s+\lambda+2\mu)\alpha_2^{i+1} - 2\mu\delta_{i0}(1-\alpha_2)^2}{s(1-\alpha_2)(2\mu+\lambda\alpha_2+s\alpha_2)}$$

$$+ \frac{2\alpha_2^{i+1} + 2\delta_{i0}\,2(1-\alpha_2)}{(1-\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2)} + \frac{(\lambda+s)\alpha_2^{i+1} - 2\mu\delta_{i0}(1-\alpha_2)}{\mu(1-\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2)} . \qquad (3.15)$$

When $k \geq 3$ $E_m^*(s)$ can be determined once expressions for $P_m^*(s)$, $0 \leq m \leq k-1$, and $E_n^*(s)$ are known.

For any k, the first two terms in the expression for $E_m^*(s)$ can be inverted analytically to give $i - k + (\lambda-k\mu)t$. This shows that when $i \geq k$ the expected number in the queue depends on the number of units waiting initially in the queue at time $t = 0$ when the first k units go into service. Further, $E_m(t)$, like $E_n(t)$, increases proportionally to the arrival rate and decreases proportionally to the rate at which units are serviced by the k servers.

The final descriptor which we develop here is the expected number of idle servers in the many-server problem, which is given by (1.22). Because this is a finite sume and $P_n(t) \leq 1$ when t is finite, the linearity of the Laplace transforms implies directly that

$$E_I^*(s) = \sum_{n=0}^{k-1} (k-n)P_n^*(s). \qquad (3.16)$$

This expression can be evaluated once the functions $P_n^*(s)$, $0 \leq n \leq k-1$, are determined as described in Chapter II. When there are two servers, we find

$$E_I^*(s) = \frac{(2\mu+\lambda+s)\alpha_2^{i+1}}{\mu(1-\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2)} - \frac{\delta_{i0}}{2\mu+s\alpha_2+\lambda\alpha_2} . \qquad (3.17)$$

The final value theorem for Laplace transforms [11] can be used to prove that when $\lambda < k\mu$ the functions $E_n(t)$, $E_m(t)$, and $E_I(t)$ are convergent and approach the

steady state values E_n, E_m, and E_I as $t \to \infty$. This is done explicitly for the case $k = 1$ and $k = 2$ in Appendix II.

We have determined here expressions for the Laplace transforms of three commonly used system descriptors. None of our expressions is particularly elegant, especially when we recall that α_2 is given by

$$\frac{\lambda+k\mu+s-\sqrt{(\lambda+k\mu+s)^2-4k\lambda\mu}}{2\lambda} .$$

Even if we could invert the Laplace transforms analytically to determine $E_n(t)$, $E_m(t)$, and $E_I(t)$, it would be too much to hope that they would be any less complicated than the expressions for $P_n(t)$. We can, however, invert the Laplace transforms numerically and evaluate these transient descriptors at various points in time. This is done in Chapter IV.

4. NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

In this chapter we determine numerically the inverse Laplace transforms of the functions $E_n^*(s)$, $E_m^*(s)$ and $E_I^*(s)$ which were derived in Chapter 3. Many algorithms to do this have been described in the literature; the method used here is that of Weeks [15]. The inverse transform is obtained as a series expansion of Laguerre functions, where the coefficients of the expansion are found by trigonometric interpolation. The recurrence relations and quadrature formulae determined by Weeks are used. This method requires a subroutine to evaluate the real and imaginary parts of each function to be inverted.

Weeks states that the best results with his method were obtained using a degree of quadrature N between 20 and 50. We have chosen to take N = 20. The rightmost singularity of the functions we are inverting here is at s = 0, so the scale factor c is 0. For most cases, we have chosen to invert functions in the interval $0 \leq t \leq 50$; then T_{max} is 50 and the associated scale factor T is 2.5.

All routines were programmed in Fortran IV on an IBM-370, and all floating point arithmetic was performed with double precision. The routine which does the inversion was tested by duplicating Weeks' first example.

Considerable round-off error is introduced at three main stages of the computation. First, to calculate the expansion coefficients requires one sine and one cosine operation, $2N^2 + 6N + 5$ additions and subtractions, $3N^2 + 8N + 11$ multiplications, $3N + 8$ divisions, and $N + 1$ calls to the $f^*(s)$ subroutine, where N is the degree of quadrature of the inversion. Once these are known, it requires $2N + 1$ additions and subtractions, $4N + 5$ multiplications and divisions, and two exponentiations to evaluate f(t) for a given t. The third problem is the subroutine to evaluate $f^*(s)$.

In testing the inversion routine our results agreed with those of Weeks to within five significant decimal digits. However, the subroutine to evaluate $f^*(s)$

in this test required only four additions, one subtraction six multiplications, and two divisions. The subroutine to evaluate the functions with which we are concerned here require significantly more computation. Each subroutine must first calculate the real and imaginary parts of α_2 and then raise α_2 to the (i+1)st power, where i is the number of units waiting initially in the system. Computing α_2 requires eight additions and subtractions, fourteen multiplications and divisions, and three square root operations. If $i \geq 1$, the exponentiation of the complex number α_2 programmed with real arithmetic requires $i^2 + 4i + 4$ multiplications and $i + 2$ additions and subtractions. To finally compute the functional value $f^*(s)$ requires a minimum of fifteen additions and subtractions, twenty multiplications, and two divisions, and a maximum of 33 additions and subtractions, 66 multiplications, and two divisions.

After performing all of these arithmetic operations, we are left with at least two but at most three significant decimal digits in the final result f(t). We feel that for many applications these rather imprecise results would be sufficient, particularly if the time unit is chosen with care.

The order in which calculations were done was found to be critical. For example, to invert $E_n^*(s)$ when there is one server, one approach would be to invert the term $\alpha_2^{i+1}/s(1-\alpha_2)$ numerically and add to that quantity the inverses of the terms i/s and $(\lambda-\mu)/s^2$, which are easily determined analytically. However, the computed value of $\mathscr{L}^{-1}\left\{\alpha_2^{i+1}/s(1-\alpha_2)\right\}$ has only two or three significant digits, and adding to it the quantity $i + (\lambda-\mu)t$ yields a result of the correct order of magnitude but with no significant digits left. Thus we determine empirically that finite-precision computers do not always respect the linearity of Laplace transforms, and it is necessary to invert each entire function numerically.

It is also necessary to evaluate the functions over a common denominator so that only one complex division is performed. This is because the real part of the denominator of each term is very close to zero in magnitude. When the computation

was not done in this order, we found that the expected number in several systems which are supposed to achieve a steady state was of rapidly increasing negative magnitude. But when the same functions were computed in the suggested order, the expected number in the systems approached steady state values nicely.

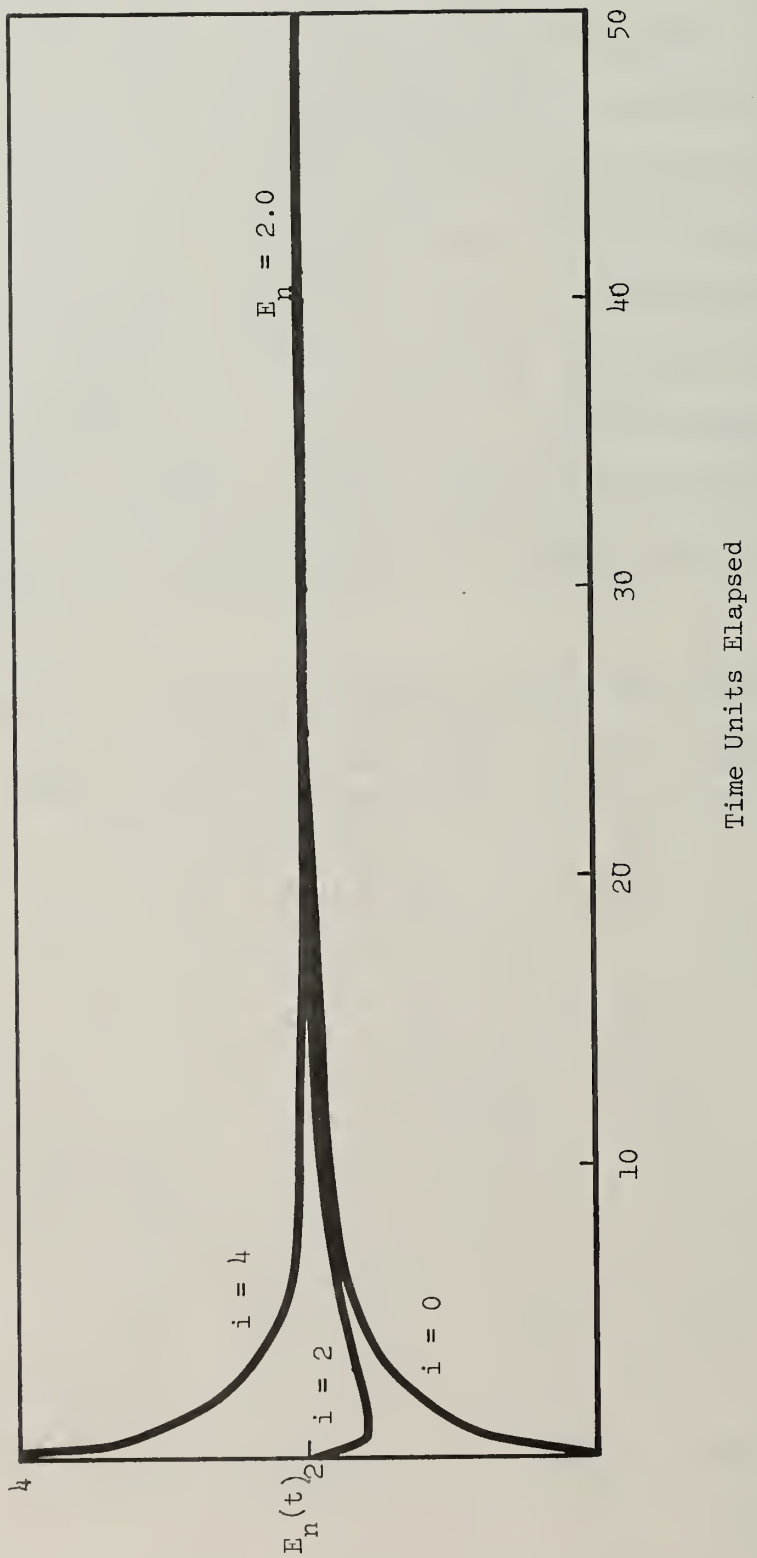
Appendix III contains tables of the numerical inversions of transient system descriptors for various different queueing systems. Recall from the introduction that those that reach steady state are characterized by the ratio $\lambda/k\mu$ being less than one. For these systems we note that the transient descriptors do indeed approach the steady state results. Moreover, this approach seems to be more rapid when $\lambda/k\mu$ is smallest and relatively slower for values of $\lambda/k\mu$ close to 1. This is as one would expect from the sensitivity analysis referred to in the introduction.

Figures 2 through 6 are graphs of the data in tables 7 through 9. These systems, for which $\lambda/k\mu = .67$, are chosen because they yield typical results for $\lambda/k\mu < 1$. The initial conditions i = 0, 2, and 4 were chosen because 2 is the value of E_n in the one-server case and close to E_n in the two-server case. Then the curves corresponding to i = 0 show $E_n(t)$ and $E_m(t)$ increasing and $E_I(t)$ decreasing while those corresponding to i = 4 show the opposite. When i = 2 a curious phenomenon is observed. Immediately after the system starts operating, $E_n(t)$ dips sharply and then climbs slowly back to the steady state value. This behavior characterizes all of the systems we investigated where the number of units in the system initially and at steady state are nearly the same.

We would next like to determine whether these systems we have studied are first order systems. When the system is empty initially, a first order system is one whose behavior can be described by the analytic function

$$E_n(t) = \nu(1-e^{-t/T})$$

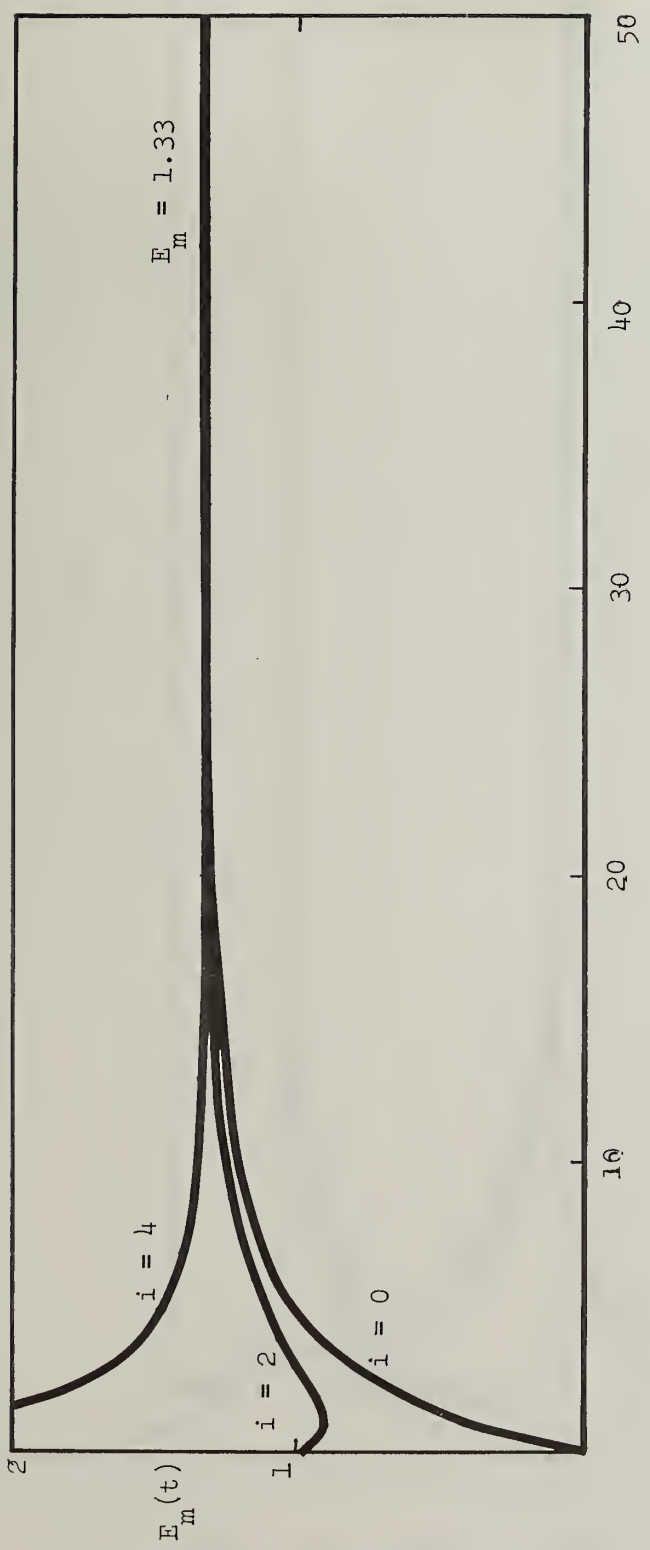
where ν is the system's final value, and T is a time constant to be determined. It



Expected Number of Units in the System

($k = 1$, $\lambda/k\mu = .67$)

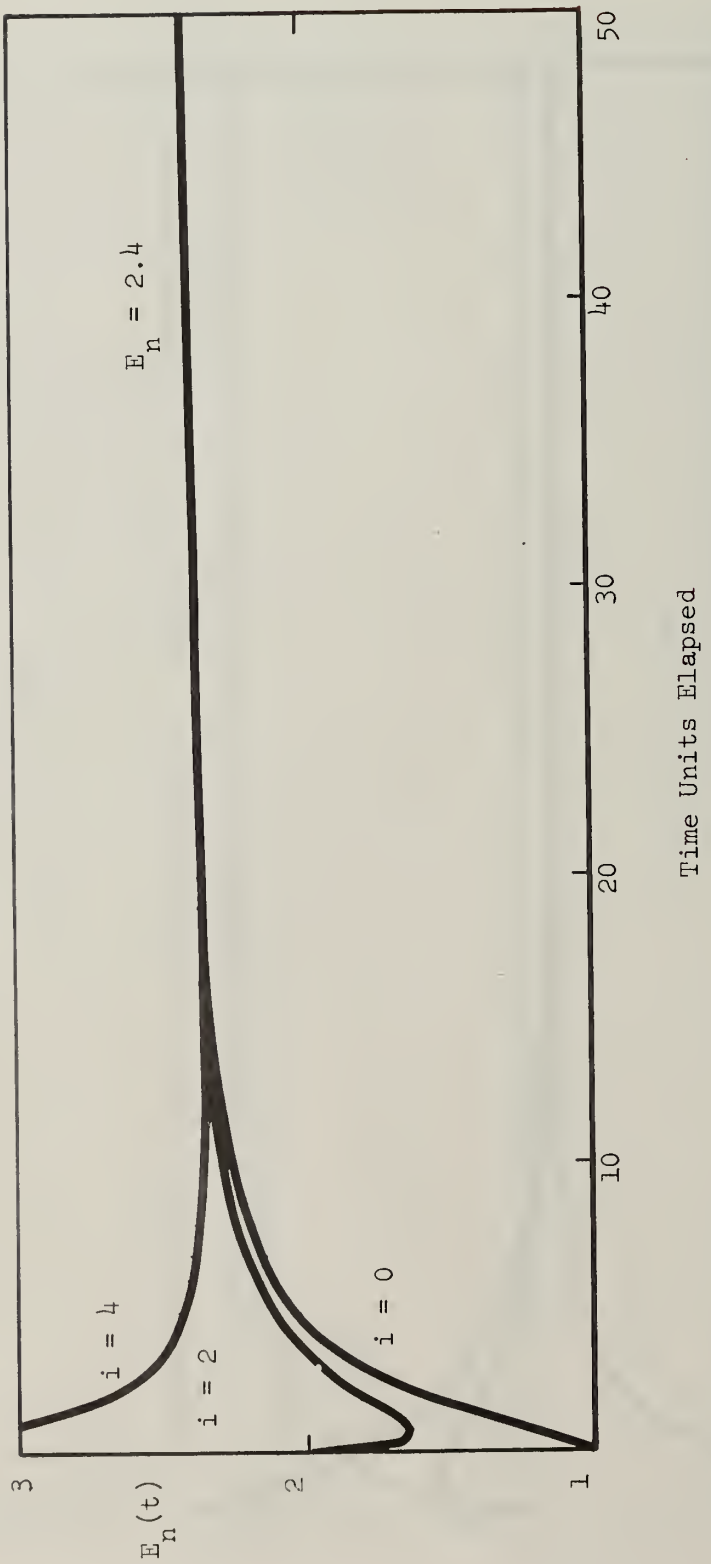
FIGURE 2



Expected Number of Units in the Queue

($k = 1$, $\lambda/k\mu = .67$)

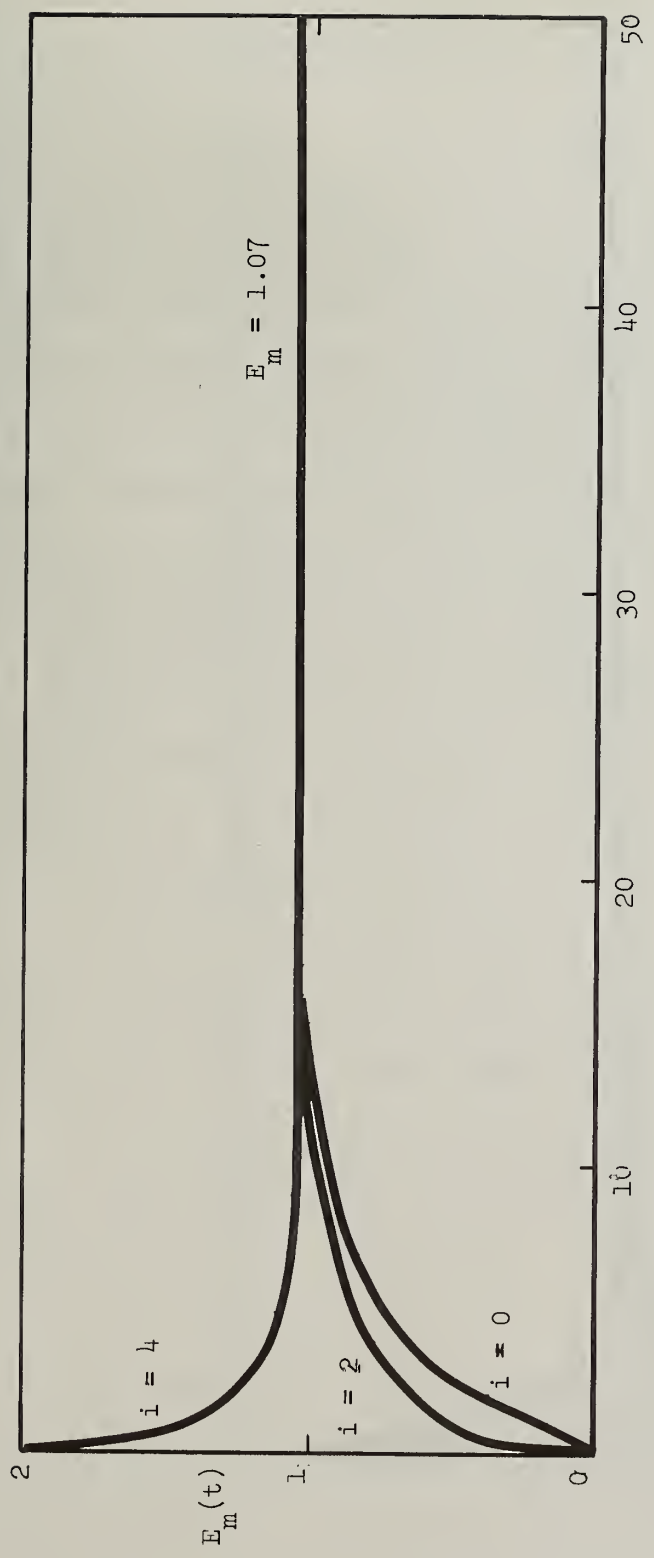
FIGURE 3



Expected Number of Units in the System

($k = 2$, $\lambda/k\mu = .67$)

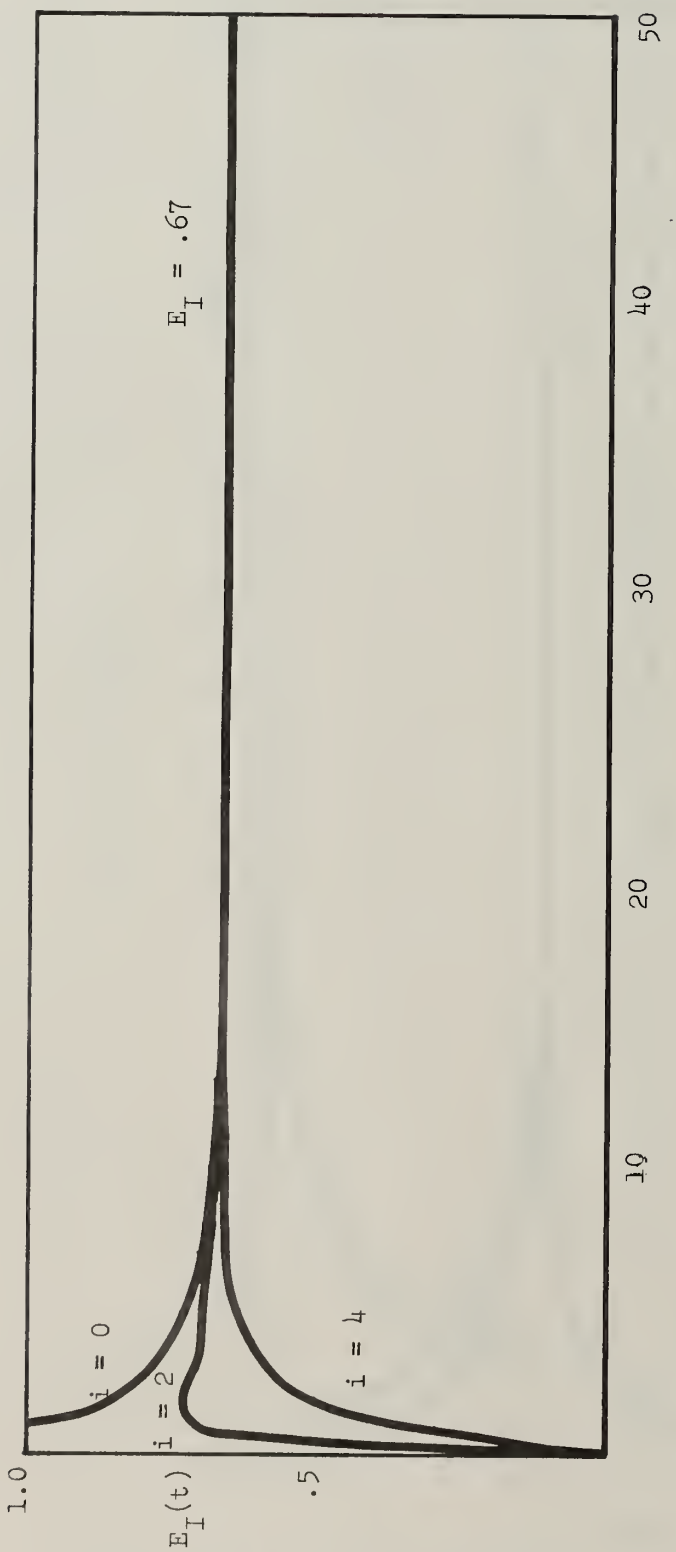
FIGURE 4



Expected Number of Units in the Queue

($k = 2$, $\lambda/k\mu = .67$)

FIGURE 5



Expected Number of Idle Servers

($k = 2$, $\lambda/k\mu = .67$)

FIGURE 6

can be trivially verified that for such a system

$$E_n(0) = 0$$

and

$$\lim_{t\to\infty} E_n(t) = \nu.$$

After one time constant has elapsed, that is when t = T, a first order system will have attained 63.2% of its final value. After two time constants it will have reached 86.5% of final value and after five time constants, 99.3% of final value.

For some of our systems which reach steady state, we approximate T by one half the time required to achieve 86.5% of final value because this can be determined more accurately than the amount of time required to achieve 63.2% of final value. We find that when $\lambda/k\mu = .50, T = 2$; when $\lambda/k\mu = .67, T = 3$; when $\lambda/k\mu = .80, T = 20$; and when $\lambda/k\mu = .909, T = 125$. In Figure 7 the percentage of final value attained is plotted against the number of time constants elapsed for the values of $\lambda/k\mu$ mentioned above when i = 0. We compare this to the function $1 - e^{-t/T}$, which is the portion of final value achieved at time t by a first order system. The curves intersect at t = 2T because of the definition used for T above. From Figure 7 we see that although the behavior of Poisson queueing systems with infinite queue length is similar to that of first order systems, they are not true first order systems. When t = T, the Poisson systems show values approximately 10% higher than would a first order system, while for $t \geq 3T$ the Poisson systems are consistently 3% to 5% lower than a first order system. Had the time constant T been chosen differently these discrepancies would have been more pronounced.

For systems which never reach steady state the functions $E_n(t)$ and $E_m(t)$ are growing steadily while $E_I(t)$ is decaying, which is as one would expect when units arrive in the system faster than they can be serviced. The tables provide a quantified measure of this growth and decay. It is illustrated in Figures 8 and 9, which show the number of units in the system and in the queue for the case i = 0 and k = 1.

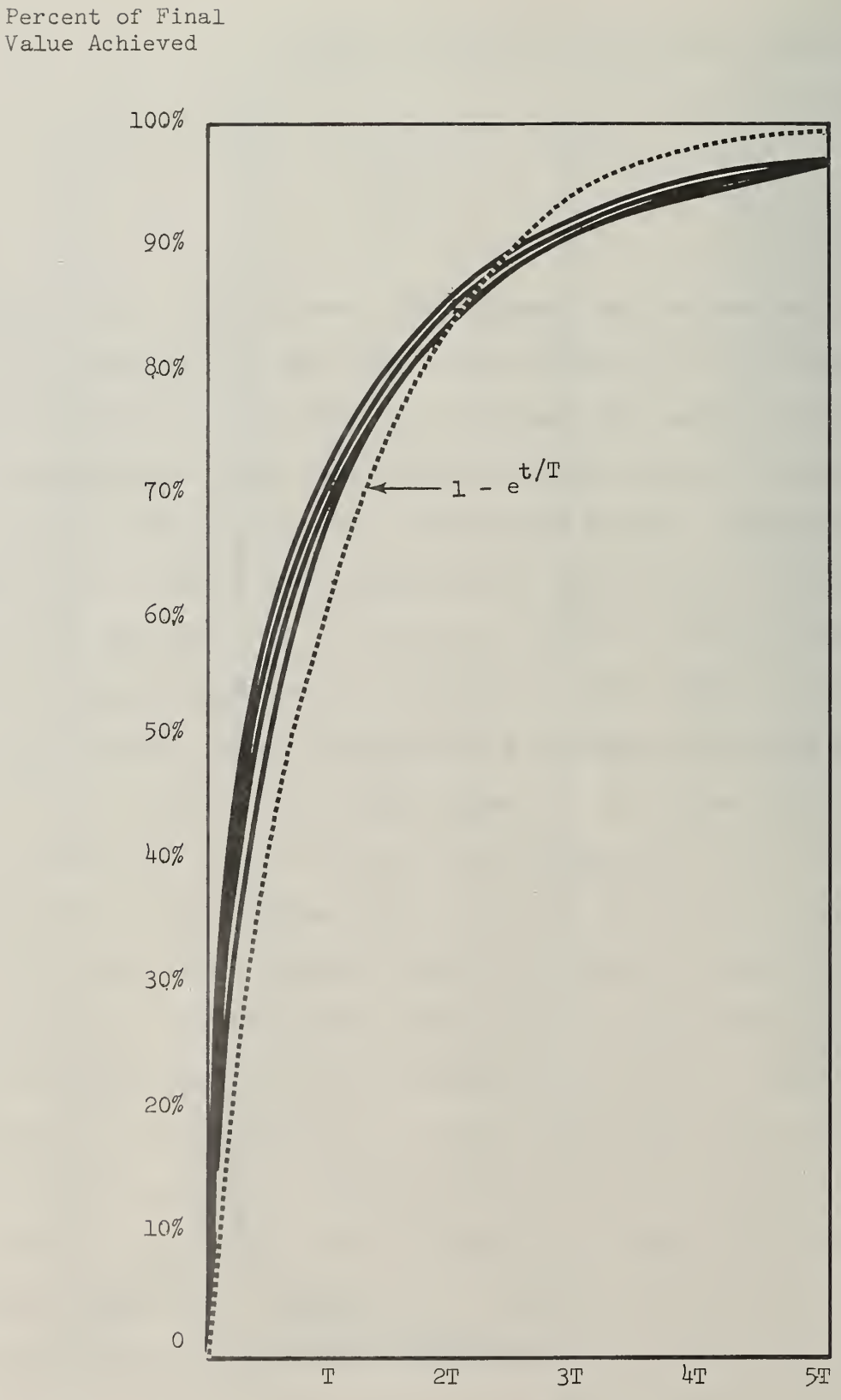
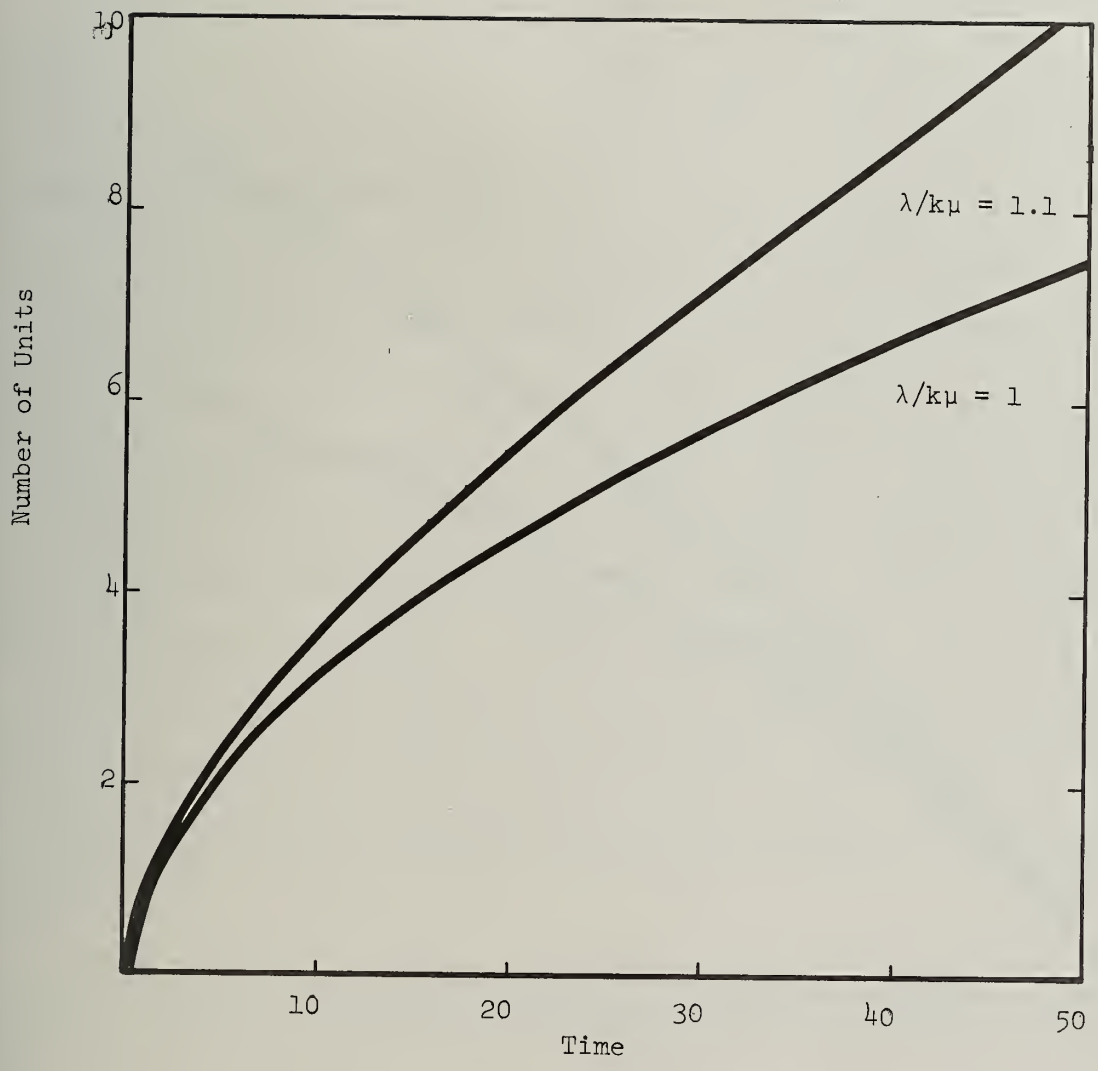
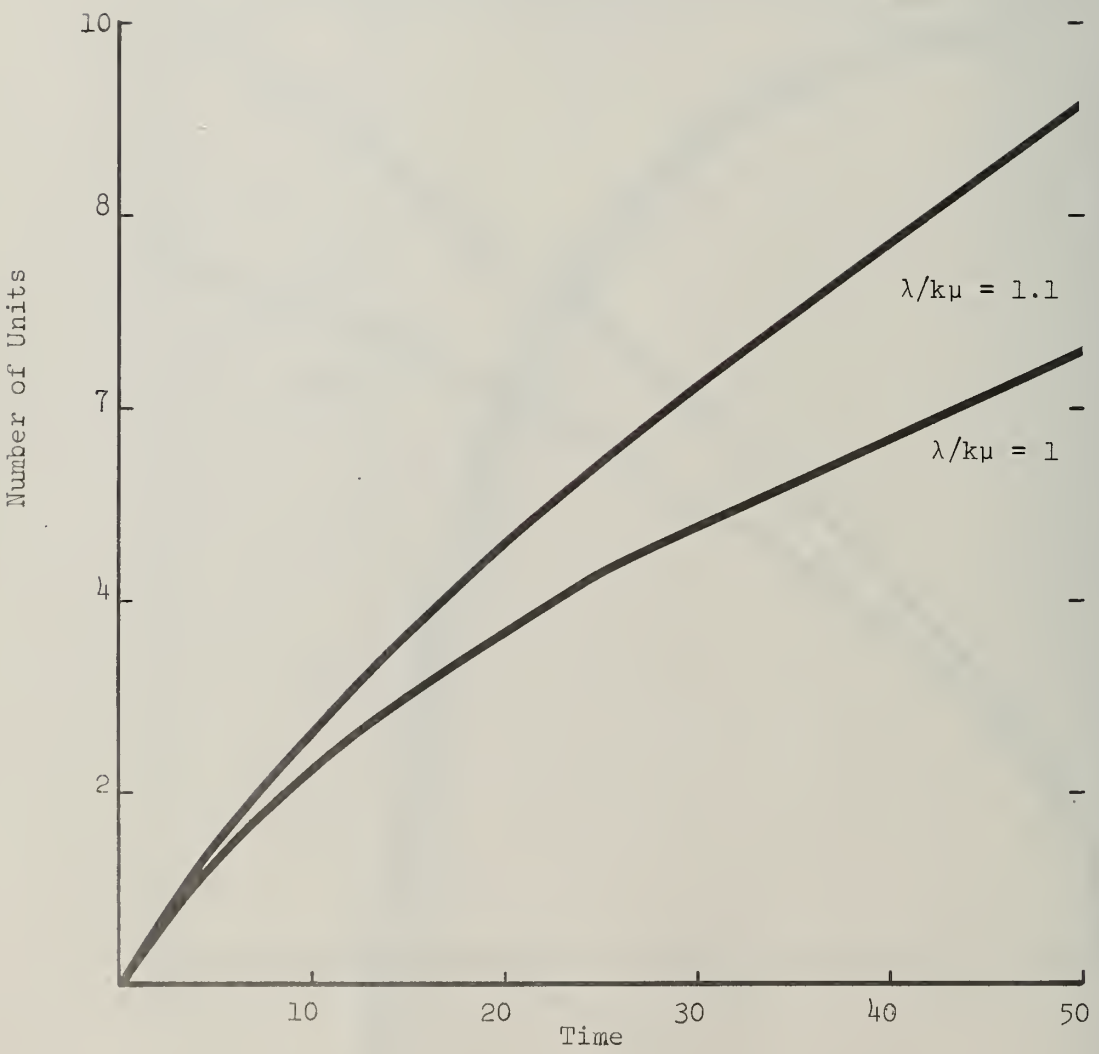

Percent of Final
Value Achieved
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0
1 - e^t/T
T
2T
3T
4T
5T
Time Constants Elapsed

FIGURE 7



Expected Number in the System $E_n(t)$

FIGURE 8



Expected Number of Units in Queue $E_m(t)$

FIGURE 9

5. CONCLUSION

In the introduction we saw that although steady state analysis of queueing systems is very useful, a transient analysis would provide more powerful information if the transient solutions could be put into a more elegant form. The discussion of time-dependent solutions in Chapter II illustrated how complicated and difficult to use they can be. In Chapter III we used theorems on the linearity and convergence of Laplace transforms to develop expressions for the Laplace transforms of some transient system descriptors. We further proved that for systems which reach steady state, the transient system descriptors approach steady state values as time increases without bound. Finally, in Chapter IV we determined the transient system descriptors $E_n(t)$, $E_m(t)$ and $E_I(t)$ by inverting their Laplace transforms numerically. Our method can be used to produce tables of values of these descriptors over arbitrary time intervals for any choices of the parameters λ, k, μ and i. We feel that these results are in a more readily accessible form than many other time-dependent results, particularly those discussed in Chapter II.

A great deal of work remains to be done. We have not considered waiting time statistics, such as the expected waiting time in the queue or the expected time in the system. There are also many other systems to which this type of analysis can be applied. One of the more important is the finite length queue, because computer systems with their limited storage space for waiting jobs are more accurately modelled by a system with finite queue length. Other applications of our work might include using tables such as those we have generated to determine analytic functions to model $E_n(t)$, $E_m(t)$ and $E_I(t)$. We saw in Chapter IV that $E_n(t)$ cannot be modelled by $\nu(1-e^{-t/T})$, which describes first order systems. Another interesting topic would be to determine the time constant T as a function of $\lambda/k\mu$. As

$\lambda/k\mu$ approaches 1, the amount of time required to reach steady state becomes infinitely large, so T must have some positive power of $(1-\lambda/k\mu)$ in its denominator. We hope that the beginning attempted here will inspire others to attempt more transient analysis of queueing systems.

APPENDIX I

TRANSIENT SOLUTION TO THE SINGLE-SERVER POISSON QUEUE

This is the method of solution given by Saaty [12]. Recall that the differential difference equations which describe the one-server case are (1.1) and (1.2). We first introduce the generating function

$$\prod(z,t) = \sum_{n=0}^{\infty} z^n P_n(t). \quad [2]$$

We multiply equation (1.1) by z and the equations (1.2) by z^{n+1} giving

$$z \frac{\partial}{\partial t} P_0(t) = - (\lambda+\mu)zP_0(t) + \mu z P_1(t) + \mu z P_0(t)$$

$$z^{n+1} \frac{\partial}{\partial t} P_n(t) = - (\lambda+\mu)z^{n+1}P_n(t) + \mu z^{n+1}P_{n+1}(t) + \lambda z^{n+1}P_{n-1}(t), \; n \geq 1.$$

Summing over all values of n we find that

$$z \frac{\partial}{\partial t} \Pi(z,t) = (1-z)\{(\mu-\lambda z)\Pi(z,t) - \mu P_0(t)\}. \qquad (I.1)$$

We next introduce the Laplace transform

$$F^*(s) = \int_0^{\infty} e^{-st} F(t)dt. \quad [11]$$

Because the Laplace transform is a linear operator, we can take the Laplace transform of the generating function to find

$$\prod^*(z,s) = \sum_{n=0}^{\infty} z^n P_n^*(s) \qquad (I.2)$$

for those values of z and s where $\Pi^*(z,s)$ is uniformly convergent.

Sufficient conditions for convergence are $|z| \leq 1$ and Re $s > 0$, because then

$$\prod^*(z,s) = \int_0^\infty \left\{ \sum_{n=0}^\infty z^n P_n(t) \right\} e^{-st} dt$$

$$\leq \int_0^\infty \left\{ \sum_{n=0}^\infty P_n(t) \right\} e^{-st} dt$$

$$= \int_0^\infty e^{-st} dt = \frac{1}{s} < \infty.$$

Taking the Laplace transform of (I.1) gives

$$z(s\Pi^*(z,s)-\Pi(z,0)) = (1-z)\{(\mu-\lambda z)\Pi^*(z,s)-\mu P_0^*(s). \tag{I.3}$$

The condition that there are i units waiting for service initially implies that $P_i(0) = 1$ and $P_n(0) = 0$ if $i \neq n$. Therefore $P_n(0) = \delta_{in}$, where δ_{in} is the Kronecker delta which is defined as

$$\delta_{ii} = 1$$

$$\delta_{in} = 0, \; i \neq n.$$

Then we find that

$$\prod(z,0) = \sum_{n=0}^\infty z^n P_n(0) = \sum_{n=0}^\infty z^n \delta_{in} = z^i.$$

Substituting this in (I.3), we solve for $\Pi^*(z,s)$ and find

$$\prod^*(z,s) = \frac{z^{i+1} - \mu(1-z)P_0^*(s)}{sz - (1-z)(\mu-\lambda z)} \tag{I.4}$$

Now we use Rouché's theorem to determine $P_0^*(s)$. The theorem states that if $f(z)$ and $g(z)$ are analytic in a simply connected domain D, C is a Jordan contour in D, and $|f(z)| > |g(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeroes inside C [9]. We apply this to the denominator of (I.4). D is the disc $|z| \leq 1$, C is the circle $|z| = 1$, $f(z) = (\lambda+\mu+s)z$ and $g(z) = \lambda z^2 + \mu$. On C, $|f(z)| = |\lambda+\mu+s| > |\lambda+\mu| = |g(z)|$ because Re $s > 0$. $f(z)$ has one zero inside C, at $z = 0$, so the denominator of (I.4) has one zero in $|z| < 1$. Now the zeroes of the denominator are

$$\alpha_1 = \frac{\lambda+\mu+s+\sqrt{(\lambda+\mu+s)^2 - 4\lambda\mu}}{2\lambda}$$

and

$$\alpha_2 = \frac{\lambda+\mu+s-\sqrt{(\lambda+\mu+s)^2 - 4\lambda\mu}}{2\lambda}$$

Since $|\alpha_2| < |\alpha_1|$, α_2 must lie in $|z| < 1$. The convergence of $\Pi^*(z,s)$ implies α_2 must also be a zero of the numerator of (I.4). Then

$$\alpha_2^{i+1} - \mu(1-\alpha_2)P_0^*(s) = 0$$

and we know that

$$P_0^*(s) = \frac{\alpha_2^{i+1}}{\mu(1-\alpha_2)} . \qquad \text{(I.5)}$$

Substituting (20) into (19) gives

$$\prod^*(z,s) = \frac{z^{i+1} - \frac{(1-z)}{(1-\alpha_2)}\alpha_2^{i+1}}{-\lambda(z-\alpha_1)(z-\alpha_2)}$$

$$= \frac{z^{i+1}(1-\alpha_2) - (1-z)\alpha_2^{i+1}}{-\lambda(z-\alpha_1)(a-\alpha_2)(1-\alpha_2)} .$$

This can be factored as

$$\prod^*(z,s) = \frac{(z-\alpha_2)\sum_{j=0}^{i} z^{i-j}\alpha_2^j - z\alpha_2(z-\alpha_2)\sum_{j=0}^{i-1} z^{i-1-j}\alpha_2^j}{\lambda\alpha_1\left(1-\frac{z}{\alpha_1}\right)(z-\alpha_2)(1-\alpha_2)}$$

Cancelling $(z-\alpha_2)$ from numerator and denominator gives

$$\prod^*(z,s) = \frac{\sum_{j=0}^{i} z^{i-j}\alpha_2^j - \alpha_2\sum_{j=0}^{i-1} z^{i-j}\alpha_2^j}{\lambda\alpha_1\left(1-\frac{z}{\alpha_1}\right)(1-\alpha_2)}$$

$$= \frac{(1-\alpha_2)\sum_{j=0}^{i} z^{i-j}\alpha_2^j + \alpha_2^{i+1}}{\lambda\alpha_1\left(1-\frac{z}{\alpha_1}\right)(1-\alpha_2)}$$

$$= \frac{1}{\lambda\alpha_1}\left(1-\frac{z}{\alpha_1}\right)^{-1}\sum_{j=0}^{i} z^{i-j}\alpha_2^j + \left(1-\frac{z}{\alpha_1}\right)^{-1}\frac{\alpha_2^{i+1}}{\lambda\alpha_1(1-\alpha_2)}. \qquad (I.6)$$

$|\alpha_1| > 1$ and $|z| \leq 1$ implies $\left|\frac{z}{\alpha_1}\right| < 1$ so

$$\left(1-\frac{z}{\alpha_1}\right)^{-1} = \sum_{\ell=0}^{\infty}\left(\frac{z}{\alpha_1}\right)^{\ell} \quad [9].$$

Substituting this into (I.6) gives

$$\prod^*(z,s) = \frac{1}{\lambda\alpha_1}\left\{\sum_{\ell=0}^{\infty}\left(\frac{z}{\alpha_1}\right)^{\ell}\right\}\left\{\sum_{j=0}^{i} z^{i-j}\alpha_2^j\right\} + \frac{\alpha_2^{i+1}}{\lambda\alpha_1(1-\alpha_2)}\left\{\sum_{\ell=0}^{\infty}\left(\frac{z}{\alpha_1}\right)^{\ell}\right\} \qquad (I.7)$$

By I.2, $P_n^*(s)$ is the coefficient of z^n in $\prod^*(z,s)$. By expanding I.7 in powers of z it can be determined that

$$P_n^*(s) = \frac{1}{\lambda}\left\{\sum_{j=0}^{i}\frac{\left(\frac{\mu}{\lambda}\right)^j}{\alpha_1^{n-i-2j+1}} + \left(\frac{\lambda}{\mu}\right)^{n+1}\sum_{j=n+i+2}^{\infty}\frac{\left(\frac{\mu}{\lambda}\right)^j}{\alpha_1^j}\right\}, \quad n\geq i$$

and

$$P_n^*(s) = \frac{1}{\lambda}\left\{\sum_{j=i-n}^{i}\frac{\left(\frac{\mu}{\lambda}\right)^j}{\alpha_1^{n-i+2j+1}} + \left(\frac{\lambda}{\mu}\right)^{n+1}\sum_{j=n+1+2}^{\infty}\frac{\left(\frac{\mu}{\lambda}\right)^j}{\alpha_1^j}\right\}, \quad n<i. \qquad (I.8)$$

To find $P_n(t)$, use the fact that

$$\mathcal{L}^{-1}\left\{\left(\frac{s+\sqrt{s^2-4\lambda\mu}}{2\lambda}\right)^{-\nu}\right\} = \frac{(2\lambda)^{\nu}\nu(2\sqrt{\lambda\mu})^{-\nu}}{t} I_{\nu}(2\sqrt{\lambda\mu}\,t)$$

$$= \frac{\nu\left(\sqrt{\frac{\lambda}{\mu}}\right)^{\nu}}{t} I_{\nu}\left(2\sqrt{\lambda\mu}\right)t \qquad (I.9)$$

where $I_{\nu}(x)$ is the modified Bessel function of the first kind, which is defined as

$$I_{\nu}(x) = \sum_{j=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{\nu+2j}}{j!\Gamma(\nu+j+1)}. \quad [11]$$

Each term of $P_n^*(s)$ includes a negative power of α_1, which looks like

$$\left\{\frac{\lambda+\mu+s+\sqrt{(\lambda+\mu+s)^2-4\lambda\mu}}{2\lambda}\right\}^{-\nu}$$

Using (I.9) and the relationship

$$f^*(s+\lambda+\mu) = e^{-(\lambda+\mu)t}f(t) \qquad [11]$$

(I.8) can be inverted as follows:

$$P_n(t) = \frac{e^{-(\lambda+\mu)t}}{\lambda}\left\{\sum_{j=0}^{i}\frac{n-i+2j+1}{t}\left(\sqrt{\frac{\lambda}{\mu}}\right)^{n-i+1} I_{n-i+2j+1}\right.$$

$$\left. + \left(\frac{\lambda}{\mu}\right)^{n+1}\sum_{j=n+i+2}^{\infty}\frac{j}{t}\left(\sqrt{\frac{\mu}{\lambda}}\right)^{j} I_j\right\}, \quad n \geq i$$

$$P_n(t) = \frac{e^{-(\lambda+\mu)t}}{\lambda}\left\{\sum_{j=i-n}^{i}\frac{n+2j-i+1}{t}\left(\sqrt{\frac{\lambda}{\mu}}\right)^{n-i+2j+1} I_{n-i+2j+1}\right.$$

$$\left. + \left(\frac{\lambda}{\mu}\right)^{n+1}\sum_{j=n+i+2}^{\infty}\frac{j}{t}\left(\sqrt{\frac{\mu}{\lambda}}\right)^{j} I_j\right\}, \quad n < i,$$

where the argument of the Bessel function is $2\sqrt{\lambda\mu}\ t$.

Finally, we can use these two facts about Bessel functions

$$\frac{2\nu}{z} I_\nu(z) = I_{\nu-1}(z) - I_{\nu+1}(z)$$

[11]

$$I_\nu(z) = I_{-\nu}(z)$$

to determine that

$$P_n(t) = e^{-(\lambda+\mu)t}\left\{\left(\sqrt{\frac{\mu}{\lambda}}\right)^{i-n} I_{n-i} + \left(\sqrt{\frac{\mu}{\lambda}}\right)^{i-n+1} I_{n+i+1}\right.$$

$$\left. + \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) \sum_{j=n+i+2}^{\infty} \left(\sqrt{\frac{\mu}{\lambda}}\right)^j I_j \right\} \qquad \text{(I.10)}$$

for all $n \geq 0$.

APPENDIX II

FINAL VALUES OF THE TRANSIENT SYSTEM DESCRIPTORS

To determine the final values of the transient system descriptors $E_n(t)$, $E_m(t)$ and $E_I(t)$, we employ the final value theorem for Laplace transforms, which states

$$\lim_{s\to 0} sf^*(s) = \lim_{t\to\infty} f(t). \quad [11]$$

When there is one server we find

$$E_n = \lim_{t\to\infty} E_n(t) = \lim_{s\to 0} sE_n^*(s)$$

$$= \lim_{s\to 0} s\left\{\frac{i}{s} + \frac{(\lambda-\mu)}{s^2} + \frac{\alpha_2^{i+1}}{s(1-\alpha_2)}\right\}$$

$$= i + \lim_{s\to 0} \frac{(\lambda-\mu)(1-\alpha_2) + s\alpha_2^{i+1}}{s(1-\alpha_2)} .$$

This is an indeterminate form of type $\frac{0}{0}$. We apply L'Hospital's Rule twice to find that this is equal to

$$i + \lim_{s\to 0} \frac{(\lambda-\mu)\left(-\frac{\partial^2\alpha_2}{\partial s^2}\right) + 2(i+1)\alpha_2^i \frac{\partial\alpha_2}{\partial s} + s(i+1)\frac{\partial}{\partial s}\left(\alpha_2^i \frac{\partial\alpha_2}{\partial s}\right)}{-2\frac{\partial\alpha_2}{\partial s} - s\frac{\partial^2\alpha_2}{\partial s^2}} .$$

The first and second partial derivatives of α_2 with respect to s are

$$\frac{\partial\alpha_2}{\partial s} = \frac{1 - ((k\mu+\lambda+s)^2 - 4k\lambda\mu)^{-\frac{1}{2}}(k\mu+\lambda+s)}{2\lambda}$$

$$\frac{\partial^2\alpha_2}{\partial s^2} = \frac{((k\mu+\lambda+s)^2 - 4k\lambda\mu)^{-3/2}(k\mu+\lambda+s)^2 - ((k\mu+\lambda+s)^2 - 4k\lambda\mu)^{-1/2}}{2\lambda} .$$

Letting s approach zero, we find

$$\lim_{s\to 0} \alpha_2 = 1$$

$$\lim_{s\to 0} \frac{\partial \alpha_2}{\partial s} = \frac{1 - (k\mu-\lambda)^{-1}(k\mu+\lambda)}{2\lambda}$$

$$\lim_{s\to 0} \frac{\partial^2 \alpha_2}{\partial s^2} = \frac{(k\mu-\lambda)^{-3}(k\mu+\lambda)^2 - (k\mu-\lambda)^{-1}}{2\lambda} .$$

Substituting these values into the expression above gives

$$\lim_{s\to 0} sE_n^*(s) = \frac{\lambda}{\mu-\lambda}$$

which is the well-known value of E_n stated in Chapter I.

When there are two servers,

$$\lim_{s\to 0} sE_n^*(s)$$

$$= i + \left\{ \lim_{s\to 0} \frac{\lambda-2\mu}{s} + \frac{s(s+\lambda+2\mu)\alpha_2^{i+1}}{(s\alpha_2+\lambda\alpha_2+2\mu)(1-\alpha_2)} - \frac{2\mu s\delta_{i0}(1-\alpha_2)}{s\alpha_2+\lambda\alpha_2+2\mu} \right\}$$

$$= i + \left\{ \lim_{s\to 0} (2\mu-\lambda)(\alpha_2-1)(s\alpha_2+\lambda\alpha_2+2\mu) + (s^2+\lambda s+2\mu)\alpha_2^{i+1} - 2\mu s\delta_{i0}(1-\alpha_2)^2 \right\} / \left\{ (s\alpha_2+\lambda\alpha_2+2\mu)(s-s\alpha_2) \right\} .$$

This limit is an indeterminate form of type $\frac{0}{0}$. We apply L'Hospital's Rule twice and find that it is

$$i + \lim_{s\to 0} \Bigg\{ (2\mu+s\alpha_2+\lambda\alpha_2)(2\mu-\lambda) \frac{\partial^2\alpha_2}{\partial s^2} + 2\left(\alpha_2 + s\frac{\partial\alpha_2}{\partial s} + \lambda\frac{\partial\alpha_2}{\partial s}\right)(2\mu-\lambda)\frac{\partial\alpha_2}{\partial s}$$

$$+ \left(s\frac{\partial^2\alpha_2}{\partial s^2} + 2\frac{\partial\alpha_2}{\partial s} + \lambda\frac{\partial^2\alpha_2}{\partial s^2}\right)(2\mu-\lambda)(\alpha_2-1)$$

$$+ (s^2+s\lambda+2\mu s)(i+1)\,\alpha_2^i\frac{\partial^2\alpha_2}{\partial s^2} + (s^2+s\lambda+2\mu s)(i+1)i\alpha_2^{i-1}\left(\frac{\partial\alpha_2}{\partial s}\right)^2$$

$$+ 2(2s+\lambda+2\mu)(i+1)\,\alpha_2^i\frac{\partial\alpha_2}{\partial s} + 2\alpha_2^{i+1}\Bigg\} \Big/ \Bigg\{ (2\mu+s\alpha_2+\lambda\alpha_2)\left(-2\frac{\partial\alpha_2}{\partial s} - s\frac{\partial^2\alpha_2}{\partial s^2}\right)$$

$$+ 2\left(\alpha_2 + s\frac{\partial\alpha_2}{\partial s} + \lambda\frac{\partial\alpha_2}{\partial s}\right)\left(1-\alpha_2 - s\frac{\partial\alpha_2}{\partial s}\right)$$

$$+ \left(2\frac{\partial\alpha_2}{\partial s} + s\frac{\partial^2\alpha_2}{\partial s^2} + \lambda\frac{\partial^2\alpha_2}{\partial s^2}\right)(s-s\alpha_2) \Bigg\}.$$

Taking the limit as s approaches 0, this becomes

$$\frac{4\lambda\mu}{4\mu^2-\lambda^2}$$

which is equivalent to the expression given for E_n in Chapter I when k = 2.

When there is one server we find that

$$E_m = \lim_{t\to\infty} E_m(t) = \lim_{s\to 0} sE_m^*(s)$$

$$= \lim_{s\to 0} i - 1 + \frac{\lambda-\mu}{s} + \frac{(s+\mu)\alpha_2^{i+1}}{\mu(1-\alpha_2)}$$

$$= i - 1 + \lim_{s\to 0} \frac{\mu(\mu-\lambda)(\alpha_2-1) + (s^2+s\mu)\alpha_2^{i+1}}{s\mu(1-\alpha_2)}.$$

This is an indeterminate form of type $\frac{0}{0}$. We apply L'Hospital's Rule twice to find that it is

$$i - 1 + \lim_{s\to 0}\left\{\mu(\mu-\lambda)\frac{\partial^2\alpha_2}{\partial s^2} + 2(2s+\mu)(i+1)\alpha_2^i\frac{\partial\alpha_2}{\partial s} + 2\alpha_2^{i+1} + (s^2+s\mu)(i+1)\frac{\partial}{\partial s}\left(\alpha_2^i\frac{\partial\alpha_2}{\partial s}\right)\right\} \Big/ \left\{-2\mu\frac{\partial\alpha_2}{\partial s} - s\mu\frac{\partial^2\alpha_2}{\partial s^2}\right\}.$$

Taking the limit as s approaches 0 this becomes

$$\frac{(\lambda/\mu)^2}{1-(\lambda/\mu)}$$

which is the steady state result given by (1.15) for E_m when k = 1.

In the two-server case we find

$$E_m = \lim_{t\to\infty} E_m(t) = \lim_{s\to 0} sE_m^*(s)$$

$$= \lim_{s\to 0}\left\{i - 2 + \frac{\lambda-2\mu}{s} + \frac{(s+\lambda+2\mu)\alpha_2^{i+1}}{(1-\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2)} - \frac{2\mu\delta_{i0}(1-\alpha_2)}{2\mu+\lambda\alpha_2+s\alpha_2} + \frac{2s\alpha_2^{i+1}}{(1-\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2)} + \frac{2s\delta_{i0}\alpha_2}{2\mu+\lambda\alpha+s\alpha_2} + \frac{s(\lambda+s)\alpha_2^{i+1}}{\mu(1-\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2)} - \frac{2s\delta_{i0}}{2\mu+\lambda\alpha_2+s\alpha_2}\right\}.$$

The limits of all terms involving δ_{i0} vanish and we are left with

$$i - 2 + \lim_{s\to 0} \Big\{ \mu(2\mu-\lambda)(\alpha_2-1)(2\mu+s\alpha_2+\lambda\alpha_2) + (3s^2\mu+s\lambda\mu+2s\mu^2+s^2\lambda+s^3)\alpha_2^{i+1} \Big\} \Big/ \Big\{ \mu(s-s\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2) \Big\} .$$

This limit is an indeterminate form of type $\frac{0}{0}$, to which we apply L'Hospital's Rule twice. It then becomes

$$\begin{aligned} i - 2 + \lim_{s\to 0} \Big\{ & \mu(2\mu-\lambda)(\alpha_2-1)\left(2\frac{\partial\alpha_2}{\partial s} + s\frac{\partial^2\alpha_2}{\partial s^2} + \lambda\frac{\partial^2\alpha_2}{\partial s^2}\right) \\ & + 2\mu(2\mu-\lambda)\frac{\partial\alpha_2}{\partial s}\left(\alpha_2 + s\frac{\partial\alpha_2}{\partial s} + \lambda\frac{\partial\alpha_2}{\partial s}\right) \\ & + \mu(2\mu-\lambda)\frac{\partial^2\alpha_2}{\partial s^2}(2\mu+s\alpha_2+\lambda\alpha_2) \\ & + (3s^2\mu+s\lambda\mu+2s\mu^2-s^2\lambda+s^3)(i+1)\frac{\partial}{\partial s}\left(\alpha_2^i\frac{\partial\alpha_2}{\partial s}\right) \\ & + 2(6s\mu+\lambda\mu+2\mu^2+2s\lambda+3s^2)(i+1)\alpha_2^i\frac{\partial\alpha_2}{\partial s} + (6\mu+2\lambda+6s)\alpha_2^{i+1} \Big\} \Big/ \\ & \Big\{ \mu(s-s\alpha_2)\left(2\frac{\partial\alpha_2}{\partial s} + s\frac{\partial^2\alpha_2}{\partial s^2} + \lambda\frac{\partial^2\alpha_2}{\partial s^2}\right) + 2\mu\left(1 - s\frac{\partial\alpha_2}{\partial s} - \alpha_2\right) \\ & \left(\alpha_2 + s\frac{\partial\alpha_2}{\partial s} + \lambda\frac{\partial\alpha_2}{\partial s}\right) + \mu\left(-2\frac{\partial\alpha_2}{\partial s} - s\frac{\partial^2\alpha_2}{\partial s^2}\right)(2\mu+s\alpha_2+\lambda\alpha_2) \Big\} . \end{aligned}$$

Taking the limit as s approaches 0, this is evaluated as

$$\frac{\lambda^3/\mu}{4\mu^2-\lambda^2}$$

which is equivalent to the steady state value E_m given by (1.16) when k = 2.

Finally, we evaluate

$$E_I = \lim_{t\to\infty} E_I(t) = \lim_{s\to 0} sE_I^*(s)$$

$$= \lim_{s\to 0} \left\{ \frac{s(2\mu+\lambda+s)\alpha_2^{i+1}}{\mu(1-\alpha_2)(2\mu+s\alpha_2+\lambda\alpha_2)} - \frac{s\delta_{i0}}{2\mu+s\alpha_2+\lambda\alpha_2} \right\}.$$

The limit of the term involving δ_{i0} vanishes. We apply L'Hospital's Rule to the first term and take the limit as s approaches 0, giving

$$2 - \lambda/\mu$$

which is the steady state value for E_I when k = 2.

APPENDIX III

RESULTS OF THE NUMERICAL INVERSION OF THE LAPLACE TRANSFORMS

This appendix contains the tabular data resulting from the numerical inversion described in Chapter IV. The following values of $\lambda/k\mu$ are used: .33, .50, .67, .80, .91, 1.0, 1.1. For $\lambda/k\mu < 1$, the values of i are chosen so that the system can be observed approaching steady state in three ways. First we see how the system accumulates units when initially empty. Then we choose an initial condition close to steady state values. Third, we observe the number of units in system and queue decreasing to steady state values. Finally, when $\lambda/k\mu \geq 1$ we see how systems which don't reach steady state grow.

TABLE 1

	λ=1		i=		$\lambda/k\mu$=.33
	μ=3	k=1	μ=1.5	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	.38	.091	.65	.025	1.41
2.0	.46	.136	.71	.054	1.38
3.0	.48	.154	.73	.070	1.36
4.0	.49	.161	.74	.077	1.35
5.0	.50	.164	.75	.080	1.35
6.0	.50	.165	.75	.082	1.35
7.0	.50	.166	.75	.083	1.34
8.0	.50	.166	.75	.083	1.34
9.0	.50	.166	.75	.083	1.34
10.0	.50	.167	.75	.083	1.34
11.0	.50	.167	.75	.083	1.34
12.0	.50	.167	.75	.083	1.34
13.0	.50	.167	.75	.083	1.34
14.0	.50	.167	.75	.083	1.33
15.0	.50	.167	.75	.083	1.33
16.0	.50	.167	.75	.083	1.33
17.0	.50	.167	.75	.083	1.33
18.0	.50	.167	.75	.083	1.33
19.0	.50	.167	.75	.083	1.33
20.0	.50	.167	.75	.083	1.33
25.0	.50	.167	.75	.083	1.33
30.0	.50	.166	.75	.083	1.33
40.0	.50	.167	.75	.083	1.33
50.0	.50	.166	.75	.083	1.33
Steady State:	.50	.167	.75	.083	1.33

TABLE 2

	λ=1		i=1		λ/kμ=.33
	μ=3	k=1	μ=1.5	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	.51	.165	.65	.061	1.29
2.0	.50	.163	.71	.076	1.33
3.0	.50	.162	.73	.080	1.33
4.0	.50	.165	.74	.082	1.33
5.0	.50	.167	.75	.083	1.33
6.0	.50	.166	.75	.083	1.33
7.0	.50	.166	.75	.083	1.33
8.0	.50	.166	.75	.083	1.33
9.0	.50	.166	.75	.083	1.33
10.0	.50	.167	.75	.083	1.33
11.0	.50	.167	.75	.083	1.33
12.0	.50	.167	.75	.083	1.33
13.0	.50	.167	.75	.083	1.33
14.0	.50	.167	.75	.083	1.33
15.0	.50	.167	.75	.083	1.33
16.0	.50	.167	.75	.083	1.33
17.0	.50	.167	.75	.083	1.33
18.0	.50	.167	.75	.083	1.33
19.0	.50	.167	.75	.083	1.33
20.0	.50	.167	.75	.083	1.33
25.0	.50	.167	.75	.083	1.33
30.0	.50	.167	.75	.083	1.33
40.0	.50	.167	.75	.083	1.33
50.0	.50	.167	.75	.083	1.33
Steady State:	.50	.167	.75	.083	1.33

TABLE 3

	λ=1		i=2		$\lambda/k\mu$=.33
	μ=3	k=1	μ=1.5	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	.86	.37	.95	.148	1.08
2.0	.61	.24	.82	.115	1.26
3.0	.55	.20	.76	.095	1.30
4.0	.52	.18	.76	.090	1.32
5.0	.51	.17	.75	.087	1.33
6.0	.50	.17	.75	.085	1.33
7.0	.50	.17	.75	.083	1.33
8.0	.50	.17	.75	.083	1.33
9.0	.50	.17	.75	.083	1.33
10.0	.50	.17	.75	.083	1.33
11.0	.50	.17	.75	.084	1.33
12.0	.50	.17	.75	.084	1.33
13.0	.50	.17	.75	.084	1.33
14.0	.50	.17	.75	.083	1.33
15.0	.50	.17	.75	.083	1.33
16.0	.50	.17	.75	.083	1.33
17.0	.50	.17	.75	.083	1.33
18.0	.50	.17	.75	.083	1.33
19.0	.50	.17	.75	.084	1.33
20.0	.50	.17	.75	.084	1.33
25.0	.50	.17	.75	.084	1.33
30.0	.50	.17	.75	.083	1.33
40.0	.50	.17	.75	.083	1.33
50.0	.50	.17	.75	.083	1.33
Steady State:	.50	.17	.75	.083	1.33

TABLE 4

	$\lambda=1$		$i=0$		$\lambda/k\mu=.5$
	$\mu=2$	$k=1$	$\mu=1$	$k=2$	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	.51	.14	.84	.039	1.26
2.0	.70	.26	1.02	.11	1.17
3.0	.79	.33	1.12	.17	1.11
4.0	.86	.38	1.18	.21	1.08
5.0	.90	.41	1.22	.25	1.06
6.0	.92	.43	1.25	.27	1.04
7.0	.94	.45	1.27	.28	1.03
8.0	.96	.46	1.29	.30	1.02
9.0	.97	.47	1.30	.30	1.02
10.0	.97	.48	1.31	.31	1.02
11.0	.98	.48	1.31	.32	1.01
12.0	.99	.49	1.32	.32	1.01
13.0	.99	.49	1.32	.32	1.01
14.0	.99	.49	1.32	.32	1.01
15.0	.99	.49	1.32	.33	1.00
16.0	.99	.49	1.33	.33	1.00
17.0	.99	.50	1.33	.33	1.00
18.0	1.00	.50	1.33	.33	1.00
19.0	1.00	.50	1.33	.33	1.00
20.0	1.00	.50	1.33	.33	1.00
25.0	1.00	.50	1.33	.33	1.00
30.0	1.00	.50	1.33	.33	1.00
40.0	1.00	.50	1.33	.33	1.00
50.0	1.00	.50	1.33	.33	1.00
Steady State:	1.00	.50	1.33	.33	1.00

TABLE 5

	λ=1		i=1		$\lambda/k\mu$=.5
	μ=2	k=1	=1	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	.77	.29	.84	.11	1.07
2.0	.82	.35	1.02	.18	1.07
3.0	.87	.39	1.12	.22	1.05
4.0	.90	.42	1.18	.25	1.04
5.0	.93	.44	1.22	.27	1.03
6.0	.95	.45	1.25	.29	1.02
7.0	.96	.46	1.27	.30	1.01
8.0	.97	.47	1.29	.31	1.01
9.0	.98	.48	1.30	.31	1.01
10.0	.98	.48	1.31	.32	1.00
11.0	.99	.49	1.31	.32	1.00
12.0	.99	.49	1.32	.32	1.00
13.0	.99	.49	1.32	.33	1.00
14.0	.99	.49	1.32	.33	1.00
15.0	.99	.49	1.32	.33	1.00
16.0	.99	.50	1.33	.33	1.00
17.0	1.00	.50	1.33	.33	1.00
18.0	1.00	.50	1.33	.33	1.00
19.0	1.00	.50	1.33	.33	1.00
20.0	1.00	.50	1.33	.33	1.00
25.0	1.00	.50	1.33	.33	1.00
30.0	1.00	.50	1.33	.33	1.00
40.0	1.00	.50	1.33	.33	1.00
50.0	1.00	.50	1.33	.33	1.00
Steady State:	1.00	.50	1.33	.33	1.00

TABLE 6

	λ=1		i=2		$\lambda/k\mu$=.5
	μ=2	k=1	μ=1	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	1.34	.67	1.31	.28	.77
2.0	1.14	.58	1.28	.30	.93
3.0	1.06	.54	1.28	.31	.97
4.0	1.03	.52	1.28	.31	.99
5.0	1.01	.51	1.29	.32	1.00
6.0	1.01	.50	1.30	.32	1.00
7.0	1.00	.50	1.30	.32	1.00
8.0	1.00	.50	1.31	.32	1.00
9.0	1.00	.50	1.32	.32	1.00
10.0	1.00	.50	1.32	.33	1.00
11.0	1.00	.50	1.32	.33	1.00
12.0	1.00	.50	1.33	.33	1.00
13.0	1.00	.50	1.33	.33	1.00
14.0	1.00	.50	1.33	.33	1.00
15.0	1.00	.50	1.33	.33	1.00
16.0	1.00	.50	1.33	.33	1.00
17.0	1.00	.50	1.33	.33	1.00
18.0	1.00	.50	1.33	.33	1.00
19.0	1.00	.50	1.33	.33	1.00
20.0	1.00	.50	1.33	.33	1.00
25.0	1.00	.50	1.33	.33	1.00
30.0	1.00	.50	1.33	.33	1.00
40.0	1.00	.50	1.33	.33	1.00
50.0	1.00	.50	1.33	.33	1.00
Steady State:	1.00	.50	1.33	.33	1.00

TABLE 7

$\lambda = 2$		$i=0$			$\lambda/k\mu = .67$
	$\mu=3$	$k=1$	$\mu=1.5$	$k=2$	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	.89	.37	1.25	.16	1.01
2.0	1.22	.63	1.59	.38	.86
3.0	1.41	.79	1.79	.54	.79
4.0	1.54	.91	1.93	.65	.76
5.0	1.64	1.00	2.03	.73	.74
6.0	1.71	1.06	2.10	.79	.72
7.0	1.76	1.11	2.15	.84	.71
8.0	1.80	1.15	2.20	.88	.70
9.0	1.83	1.18	2.23	.91	.69
10.0	1.86	1.20	2.26	.94	.69
11.0	1.89	1.23	2.28	.96	.68
12.0	1.90	1.24	2.30	.98	.68
13.0	1.92	1.26	2.32	.99	.68
14.0	1.93	1.27	2.33	1.00	.68
15.0	1.94	1.28	2.34	1.01	.67
16.0	1.95	1.28	2.35	1.02	.67
17.0	1.95	1.29	2.35	1.02	.67
18.0	1.96	1.30	2.36	1.03	.67
19.0	1.97	1.30	2.37	1.04	.67
20.0	1.97	1.31	2.37	1.04	.67
25.0	1.98	1.32	2.38	1.05	.67
30.0	1.99	1.33	2.39	1.06	.67
40.0	2.00	1.33	2.39	1.06	.67
50.0	2.00	1.33	2.40	1.06	.67
Steady State:	2.00	1.33	2.40	1.07	.67

TABLE 8

	λ=2		i=2		λ/kμ=.67
	μ=3	k=1	μ=1.5	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	1.59	.90	1.64	.50	.71
2.0	1.61	.96	1.81	.62	.74
3.0	1.67	1.02	1.93	.70	.72
4.0	1.72	1.07	2.03	.77	.71
5.0	1.77	1.12	2.10	.83	.71
6.0	1.81	1.15	2.15	.87	.70
7.0	1.84	1.18	2.20	.90	.69
8.0	1.86	1.21	2.23	.93	.68
9.0	1.89	1.23	2.26	.95	.68
10.0	1.90	1.24	2.28	.97	.68
11.0	1.92	1.26	2.30	.98	.68
12.0	1.93	1.27	2.32	1.00	.68
13.0	1.94	1.28	2.33	1.01	.67
14.0	1.95	1.28	2.34	1.01	.67
15.0	1.96	1.29	2.35	1.02	.67
16.0	1.96	1.30	2.35	1.03	.67
17.0	1.97	1.30	2.36	1.03	.67
18.0	1.97	1.31	2.37	1.04	.67
19.0	1.98	1.31	2.37	1.04	.67
20.0	1.98	1.31	2.38	1.05	.67
25.0	1.99	1.32	2.38	1.05	.67
30.0	1.99	1.33	2.39	1.06	.67
40.0	1.99	1.33	2.39	1.06	.67
50.0	2.00	1.33	2.40	1.06	.67
Steady State:	2.00	1.33	2.40	1.07	.67

TABLE 9

λ=2			i=4		$\lambda/k\mu$=.67
	μ=3	k=1	μ=1.5		k=2
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	3.12	2.23	3.04	1.47	.29
2.0	2.64	1.87	2.72	1.29	.50
3.0	2.40	1.67	2.57	1.20	.58
4.0	2.26	1.56	2.49	1.14	.62
5.0	2.18	1.49	2.45	1.11	.64
6.0	2.13	1.44	2.42	1.09	.65
7.0	2.09	1.41	2.41	1.08	.66
8.0	2.06	1.39	2.40	1.07	.66
9.0	2.05	1.38	2.39	1.07	.66
10.0	2.04	1.37	2.39	1.06	.66
11.0	2.03	1.36	2.39	1.06	.66
12.0	2.02	1.35	2.39	1.06	.67
13.0	2.01	1.35	2.39	1.06	.67
14.0	2.01	1.34	2.39	1.06	.67
15.0	2.01	1.34	2.39	1.06	.67
16.0	2.00	1.34	2.39	1.06	.67
17.0	2.00	1.34	2.39	1.06	.67
18.0	2.00	1.34	2.39	1.06	.67
19.0	2.00	1.34	2.39	1.06	.67
20.0	2.00	1.34	2.39	1.06	.67
25.0	2.00	1.33	2.40	1.06	.67
30.0	2.00	1.33	2.40	1.06	.67
40.0	2.00	1.33	2.40	1.06	.67
50.0	2.00	1.33	2.40	1.07	.67
Steady State:	2.00	1.33	2.40	1.07	.67

TABLE 10

λ=1		i=0	λ/kμ=.80		
	μ=1.25	k=1	μ=.625	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
10	2.23	1.50	2.66	1.16	.55
20	2.85	2.08	3.28	1.73	.48
30	3.18	2.40	3.61	2.05	.45
40	3.40	2.62	3.84	2.27	.43
50	3.56	2.77	4.00	2.41	.42
60	3.66	2.87	4.10	2.51	.42
70	3.73	2.94	4.17	2.58	.41
80	3.78	2.99	4.23	2.64	.41
90	3.83	3.04	4.27	2.68	.41
100	3.87	3.07	4.31	2.72	.41
110	3.90	3.10	4.34	2.74	.40
120	3.92	3.12	4.36	2.76	.40
130	3.93	3.13	4.37	2.78	.40
140	3.94	3.14	4.38	2.79	.40
150	3.95	3.15	4.39	2.80	.40
160	3.96	3.16	4.40	2.80	.40
170	3.97	3.17	4.41	2.81	.40
180	3.97	3.17	4.42	2.82	.40
190	3.98	3.18	4.42	2.82	.40
200	3.98	3.18	4.43	2.83	.40
250	3.99	3.19	4.43	2.83	.40
300	4.00	3.20	4.44	2.84	.40
400	3.99	3.19	4.44	2.84	.40
500	4.00	3.20	4.44	2.84	.40
Steady State:	4.00	3.20	4.44	2.84	.40

TABLE 11

λ			i=4		$\lambda/k\mu$=.80
	μ=1.25	k=1	μ=.625	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
10	3.22	2.43	3.41	1.90	.44
20	3.36	2.58	3.67	2.13	.43
30	3.51	2.72	3.87	2.31	.42
40	3.62	2.83	4.00	2.43	.42
50	3.70	2.91	4.11	2.53	.42
60	3.77	2.97	4.18	2.60	.41
70	3.81	3.02	4.24	2.65	.41
80	3.85	3.05	4.28	2.69	.40
90	3.88	3.08	4.31	2.72	.40
100	3.90	3.11	4.34	2.74	.40
110	3.92	3.12	4.36	2.76	.40
120	3.94	3.14	4.38	2.78	.40
130	3.95	3.15	4.39	2.79	.40
140	3.96	3.16	4.40	2.80	.40
150	3.96	3.16	4.40	2.81	.40
160	3.97	3.17	4.41	2.81	.40
170	3.97	3.17	4.42	2.82	.40
180	3.98	3.18	4.42	2.82	.40
190	3.98	3.18	4.43	2.83	.40
200	3.99	3.19	4.43	2.83	.40
250	3.99	3.19	4.43	2.84	.40
300	4.00	3.20	4.44	2.84	.40
400	3.99	3.19	4.44	2.84	.40
500	3.99	3.20	4.44	2.84	.40
Steady State:	4.00	3.20	4.44	2.84	.40

TABLE 12

	λ=1		i=8		λ/kμ=.80
	μ=1.25	k=1	μ=.625	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
10	5.91	5.00	5.95	4.19	.20
20	4.99	4.14	5.17	3.51	.32
30	4.57	3.75	4.83	3.21	.37
40	4.36	3.54	4.67	3.06	.38
50	4.24	3.43	4.59	2.98	.39
60	4.16	3.35	4.53	2.93	.39
70	4.10	3.30	4.50	2.90	.40
80	4.07	3.27	4.47	2.88	.40
90	4.05	3.25	4.46	2.87	.40
100	4.04	3.24	4.46	2.86	.40
110	4.03	3.23	4.45	2.85	.40
120	4.02	3.22	4.45	2.85	.40
130	4.01	3.21	4.45	2.85	.40
140	4.01	3.21	4.44	2.84	.40
150	4.00	3.20	4.44	2.84	.40
160	4.00	3.20	4.44	2.84	.40
170	4.00	3.20	4.44	2.84	.40
180	4.00	3.20	4.44	2.84	.40
190	4.00	3.20	4.44	2.84	.40
200	4.00	3.20	4.44	2.84	.40
250	3.99	3.20	4.44	2.84	.40
300	4.00	3.20	4.44	2.84	.40
400	3.99	3.19	4.44	2.84	.40
500	3.99	3.20	4.44	2.84	.40
Steady State:	4.00	3.20	4.44	2.84	.40

TABLE 13

	λ=1		i=0		λ/kμ=.909
	μ=1.1	k=1	μ=.55	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
20	3.8	2.9	4.2	2.5	.34
40	5.1	4.2	5.5	3.8	.28
60	5.9	5.0	6.3	4.6	.26
80	6.5	5.6	7.0	5.2	.23
100	7.0	6.1	7.4	5.7	.22
120	7.3	6.5	7.8	6.0	.21
140	7.6	6.8	8.1	6.3	.21
160	7.9	7.0	8.4	6.6	.21
180	8.1	7.2	8.6	6.8	.20
200	8.3	7.4	8.8	7.0	.20
220	8.5	7.6	9.0	7.2	.20
240	8.6	7.7	9.1	7.3	.19
260	8.8	7.9	9.2	7.4	.19
280	8.9	8.0	9.4	7.5	.19
300	9.0	8.1	9.4	7.6	.19
320	9.1	8.2	9.5	7.7	.19
340	9.1	8.2	9.6	7.8	.19
360	9.2	8.3	9.7	7.9	.19
380	9.3	8.4	9.8	8.0	.19
400	9.4	8.4	9.8	8.0	.19
500	9.6	8.7	10.0	8.2	.19
600	9.7	8.8	10.2	8.4	.18
800	9.8	8.9	10.3	8.5	.18
1000	9.9	9.0	10.4	8.6	.18
Steady State:	10.0	9.1	10.5	8.7	.18

TABLE 14

	λ=1		i=10		λ/kμ=.909
	μ=1.1	k=1	μ.55	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
20	8.5	7.6	8.6	6.7	.13
40	8.2	7.3	8.3	6.6	.18
60	8.2	7.3	8.4	6.6	.19
80	8.3	7.4	8.6	6.8	.20
100	8.4	7.5	8.7	6.9	.20
120	8.5	7.6	8.9	7.1	.20
140	8.6	7.7	9.0	7.2	.19
160	8.8	7.9	9.1	7.3	.19
180	8.9	8.0	9.3	7.5	.19
200	9.0	8.1	9.4	7.6	.19
220	9.1	8.1	9.5	7.7	.19
240	9.1	8.2	9.6	7.7	.19
260	9.2	8.3	9.6	7.8	.19
280	9.3	8.4	9.7	7.9	.19
300	9.3	8.4	9.8	7.9	.19
320	9.4	8.5	9.8	8.0	.19
340	9.4	8.5	9.9	8.1	.19
360	9.5	8.6	9.9	8.1	.19
380	9.5	8.6	10.0	8.2	.19
400	9.6	8.6	10.0	8.2	.19
500	9.7	8.8	10.1	8.3	.18
600	9.8	8.9	10.3	8.4	.18
800	9.9	9.0	10.3	8.5	.18
1000	9.9	9.0	10.4	8.6	.18
Steady State:	10.0	9.1	10.5	8.7	.18

TABLE 15

	λ=1		i=20		$\lambda/k\mu$=.909
	μ=1.1	k=1	μ=5.5	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
20	18.0	17.0	18.0	16.0	.006
40	16.2	15.3	16.2	14.3	.043
60	14.9	13.9	14.9	13.0	.081
80	13.9	12.9	14.0	12.1	.11
100	13.1	12.2	13.3	11.4	.13
120	12.6	11.6	12.8	10.9	.14
140	12.1	11.2	12.4	10.5	.15
160	11.8	10.9	12.0	10.2	.16
180	11.5	10.6	11.8	10.0	.16
200	11.3	10.4	11.6	9.8	.17
220	11.1	10.2	11.4	9.6	.17
240	11.0	10.0	11.3	9.5	.17
260	10.8	9.9	11.2	9.4	.17
280	10.7	9.8	11.1	9.3	.17
300	10.6	9.7	11.0	9.2	.17
320	10.5	9.6	10.9	9.1	.18
340	10.5	9.6	10.9	9.1	.18
360	10.4	9.5	10.8	9.0	.18
380	10.4	9.5	10.8	9.0	.18
400	10.3	9.4	10.8	8.9	.18
500	10.2	9.3	10.6	8.8	.18
600	10.1	9.2	10.6	8.7	.18
800	10.0	9.1	10.5	8.7	.18
1000	10.0	9.1	10.5	8.7	.18
Steady State:	10.0	9.1	10.5	8.7	.18

TABLE 16

	$\lambda=1$		$i=0$		$\lambda/k\mu=1$
	$\mu=1$	$k=1$	$\mu=.5$	$k=2$	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	.70	.23	1.12	.066	1.09
2.0	1.15	.53	1.56	.24	.87
3.0	1.48	.80	1.90	.45	.72
4.0	1.79	1.06	2.21	.68	.63
5.0	2.06	1.31	2.50	.90	.56
6.0	2.29	1.52	2.73	1.10	.50
7.0	2.50	1.71	2.95	1.28	.46
8.0	2.70	1.90	3.15	1.46	.43
9.0	2.89	2.08	3.35	1.64	.40
10.0	3.08	2.26	3.54	1.82	.38
11.0	3.26	2.43	3.72	1.98	.36
12.0	3.43	2.59	3.89	2.14	.34
13.0	3.58	2.74	4.04	2.28	.32
14.0	3.72	2.88	4.19	2.42	.31
15.0	3.87	3.01	4.33	2.56	.30
16.0	4.01	3.15	4.47	2.69	.29
17.0	4.15	3.29	4.62	2.83	.28
18.0	4.29	3.43	4.76	2.96	.27
19.0	4.43	3.56	4.90	3.09	.26
20.0	4.56	3.68	5.03	3.22	.26
25.0	5.13	4.24	5.60	3.78	.23
30.0	5.68	4.79	6.16	4.31	.21
40.0	6.61	5.70	7.09	5.23	.18
50.0	7.46	6.54	7.94	6.06	.16

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TABLE 17

	λ=1		i=0		$\lambda/k\mu$=1.1
	μ=.9	k=1	μ=.45	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	.73	.24	1.15	.070	1.07
2.0	1.21	.57	1.63	.26	.84
3.0	1.57	.87	2.01	.49	.68
4.0	1.93	1.18	2.37	.76	.58
5.0	2.25	1.47	2.70	1.02	.50
6.0	2.53	1.73	2.98	1.26	.44
7.0	2.78	1.96	3.24	1.48	.40
8.0	3.02	2.19	3.48	1.71	.36
9.0	3.27	2.42	3.73	1.93	.33
10.0	3.51	2.66	3.98	2.16	.31
11.0	3.75	2.88	4.21	2.38	.29
12.0	3.96	3.09	4.43	2.58	.27
13.0	4.16	3.28	4.63	2.77	.25
14.0	4.35	3.46	4.82	2.96	.24
15.0	4.54	3.65	5.02	3.14	.23
16.0	4.74	3.84	5.21	3.33	.22
17.0	4.93	4.03	5.41	3.52	.21
18.0	5.13	4.22	5.61	3.71	.20
19.0	5.32	4.41	5.80	3.90	.19
20.0	5.51	4.59	5.98	4.07	.18
25.0	6.33	5.41	6.81	4.88	.15
30.0	7.17	6.23	7.65	5.70	.13
40.0	8.63	7.69	9.12	7.15	.10
50.0	10.10	9.10	10.54	8.56	.084

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TABLE 18

	λ=1		i=2		λ/kμ=1.1
	μ=.9	k=1	μ=.45	k=2	
Time	$E_n(t)$	$E_m(t)$	$E_n(t)$	$E_m(t)$	$E_I(t)$
1.0	2.14	1.25	.58	.57	.36
2.0	2.36	1.51	.75	.88	.40
3.0	2.58	1.74	.94	1.11	.39
4.0	2.83	1.99	1.18	1.38	.37
5.0	3.09	2.24	1.43	1.63	.34
6.0	3.31	2.45	1.64	1.85	.32
7.0	3.52	2.65	1.84	2.05	.30
8.0	3.73	2.86	2.05	2.25	.28
9.0	3.94	3.07	2.26	2.46	.26
10.0	4.16	3.28	2.48	2.68	.25
11.0	4.38	3.49	2.69	2.88	.24
12.0	4.57	3.68	2.89	3.07	.23
13.0	4.75	3.86	3.07	3.25	.22
14.0	4.93	4.03	3.25	3.43	.21
15.0	5.11	4.20	3.43	3.60	.20
16.0	5.29	4.38	3.61	3.78	.19
17.0	5.48	4.56	3.80	3.96	.18
18.0	5.66	4.75	3.98	4.15	.17
19.0	5.85	4.93	4.17	4.33	.17
20.0	6.02	5.10	4.34	4.50	.16
25.0	6.81	5.88	5.14	5.28	.14
30.0	7.62	6.68	5.96	6.08	.12
40.0	9.05	8.10	7.41	7.51	.10
50.0	10.45	9.49	8.82	8.90	.081

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Supplementary Notes

Abstracts

The theory of queueing systems at steady state is more advanced than that of systems at arbitrary points in time for two reasons. First, many systems are assumed to approximate steady state after a relatively short period of time. Second, steady state analysis is easier to do than time-dependent analysis. While steady state analysis provides much useful information about queueing systems, it gives no indication of how long it takes a system to reach steady state or how the system behaves before reaching it. Further, many systems never reach steady state, so steady state analysis cannot be applied at all in these cases. In this paper we define time-dependent system descriptors which are analogous to some well-known steady state descriptors for a Poisson system with infinite queue length. We discuss some transient solutions for these systems and use them to determine the Laplace transforms of the transient descriptors we have defined. Finally, we determine the transient descriptors explicitly by inverting their Laplace transforms numerically.

Key Words and Document Analysis. 17a. Descriptors

b. Identifiers/Open-Ended Terms

c. COSATI Field/Group

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